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Computation of Time-Domain Frequency Stability and Jitter from PM Noise Measurements*

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Abstract

This paper explores the effect of phase modulation (PM), amplitude modulation (AM), and thermal noise on the rf spectrum, phase jitter, timing jitter, and frequency stability of precision sources.

1. Introduction

In this paper we review the basic definitions generally used to describe phase modulation (PM) noise, amplitude modulation (AM) noise, fractional frequency stability, timing jitter and phase jitter in precision sources. From these basic definitions we can then compute the effect of frequency multiplication or division on these measures of performance. We find that under ideal frequency multiplication or division by a factor N, the PM noise and phase jitter of a source is intrinsically changed by a factor of N^2. The fractional frequency stability and timing jitter are, however, unchanged as long as we can determine the average zero crossings. After a sufficiently large N, the carrier power density is less than the PM noise power. This condition is often referred to as carrier collapse. Ideal frequency translation results in the addition of the PM noise of the two sources. The effect of AM noise on the multiplied or translated signals can be increased or decreased depending on the component non-linearity. Noise added to a precision signal results in equal amounts of PM and AM noise. The upper and lower PM (or AM) sidebands are exactly equal and 100% correlated, independent of whether the PM (or AM) originates from random or coherent processes [1].

2. Basic Definitions

2.1 Descriptions of Voltage Wave Form

The output of a precision source can be written as

$$V(t) = [V_o + \varepsilon(t)][\cos(2\pi\nu_o + \phi(t))], \qquad (1)$$

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where υ_0 is the average frequency, and V_0 is the average amplitude. Phase/frequency variations are included in $\phi(t)$ and the amplitude variations are included in $\varepsilon(t)$ [2]. The instantaneous frequency is given by

$$\nu = \nu_0 + \frac{1}{2\pi}\frac{d}{dt}\phi(t) \qquad (2a)$$

The instantaneous fractional frequency deviation is given by

$$y(t) = \frac{1}{2\pi\nu_0}\frac{d}{dt}\phi(t) \qquad (2b)$$

The power spectral density (PSD) of phase fluctuations $S_\phi(f)$ is the mean squared phase fluctuation $\delta\phi(f)$ at Fourier frequency f from the carrier in a measurement bandwidth of 1 Hz. This includes the contributions at both the upper and lower sidebands. These sidebands are exactly equal in amplitude and are 100% correlated [1]. Thus experimentally

$$S_\phi(f) = \frac{[\delta\phi(f)]^2}{BW} \qquad \text{radians}^2/\text{Hz}, \qquad (3)$$

$$BW \ll f,\ 0 < f < \infty,$$

where BW is the measurement bandwidth in Hz. Since the BW is small compared to f, $S_\phi(f)$ appears locally to be white and obeys Gaussian statistics. The fractional 1-sigma confidence interval is $1 \pm 1/N^{1/2}$ [3].

Often the PM noise is specified as single side band noise £(f), which is defined as ½ of $S_\phi(f)$. The units are generally given in dBc/Hz, which is short hand for dB below the carrier in a 1 Hz bandwidth.

$$£(f) = 10 \log [\tfrac{1}{2} S_\phi(f)] \qquad \text{dBc/Hz}. \qquad (4)$$

Frequency modulation noise is often specified as $S_y(f)$ which is the PSD of fractional frequency fluctuations. $S_y(f)$ is related to $S_\phi(f)$ by

$$S_y(f) = \frac{f^2}{\nu^2} S_\phi(f) \qquad 1/\text{Hz}. \qquad (5)$$

In the laser literature one often sees the frequency noise expressed as the PSD of frequency modulation $S_{\dot\phi}$, which is related to $S_y(f)$ as.

$$S_{\dot\phi}\,\phi(f) = \nu^2 S_y(f) = f^2 S_\phi(f) \qquad \text{Hz}^2/\text{Hz}. \qquad (6)$$

The amplitude modulation (AM) noise $S_a(f)$ is the mean squared fractional amplitude fluctuations at Fourier frequency f from the carrier in a measurement bandwidth of 1 Hz. Thus experimentally

$$S_a(f) = \left(\frac{\delta\varepsilon(f)}{V_0}\right)^2 \frac{1}{BW} \qquad 1/\text{Hz}, \qquad (7)$$

$$BW \ll f,\ 0 < f < \infty,$$

where BW is the measurement bandwidth in Hz.

The rf power spectrum for small PM and AM noise is approximately given by

$$V^2(f) \cong V_o^2[e^{-\phi_c^2} + S_\phi(f) + S_a(f)] \tag{8}$$

Where $e^{-\phi_c^2}$ is the approximate power in the carrier at Fourier frequencies from 0 to f_c. ϕ_c^2 is the mean squared phase fluctuation due to the PM noise at frequencies larger than f_c [4]. ϕ_c^2 is calculated from.

$$\phi_c^2 = \int_{f_c}^{\infty} S_\phi(f)df. \tag{9}$$

The half-power bandwidth of the signal, 2 fc can be found by setting $\phi_c^2 = 0.7$. The difference between the half-power and the 3 dB bandwidth depends on the shape of $S_\phi(f)$ [4].

2.2 Frequency Stability In The Time Domain

The frequency of even a precision source is often not stationary in time, so traditional statistical methods to characterize it diverge with increasing number of samples [2]. Special statistics have been developed to handle this problem. The most common is the two-sample or Allan variance (AVAR), which is based on analyzing the fluctuations of adjacent samples of fractional frequency averaged over a period τ. The square root of the Allan variance $\sigma_y(\tau)$, often called ADEV, is defined as

$$\sigma_y(\tau) = \left\langle \frac{1}{2}[\bar{y}(t+\tau) - \bar{y}(t)]^2 \right\rangle^{1/2} \tag{10}$$

$\sigma_y(\tau)$ can be estimated from a finite set of frequency averages, each of length τ from

$$\sigma_y(\tau) = \left[\frac{1}{2(M-1)} \sum_{i=1}^{M-1} (y_{i-1} - y_i)^2\right]^{1/2} \tag{11}$$

This assumes that there is no dead time between samples [2]. If there is dead time, the results are biased depending on the amount of dead time and the type of PM noise. See [2] for details.

$\sigma_y(\tau)$ can also be calculated from the $S_\phi(f)$ using

$$\sigma_y(\tau) = \left(\frac{\sqrt{2}}{\pi \nu_o \tau}\right)\left[\int_0^{\infty} H_\phi(f)[S_\phi(f)]\sin^4(\pi f \tau)df\right]^{1/2} \tag{12}$$

where $H_\phi(f)$ is the transfer function of the system used for measuring $\sigma_y(\tau)$ or δt below [2]. $H_\phi(f)$ must

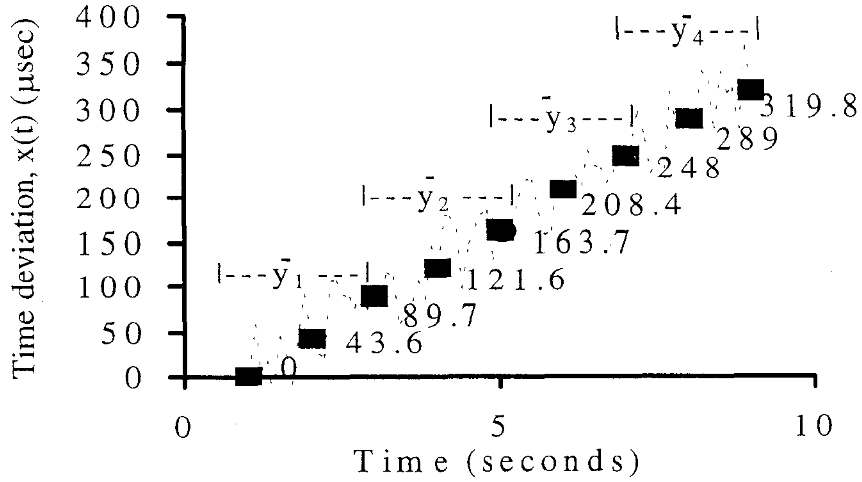

Figure 1. Placement of the y_{is} used in the computation of $\sigma_y(\tau)$ and $\delta t = \tau\sigma_y(\tau)$.

have a low-pass characteristic for $\sigma_y(\tau)$ to converge in the presence of white PM or flicker PM noise. In practice the measurement system always has a finite bandwidth but if this is not controlled or known, the results for $\sigma_y(\tau)$ will have little meaning [2]. See Table 1. If $H_\phi(f)$ has a low pass characteristic with a very sharp roll off at a maximum frequency f_h, it can be replaced by 1 and the integration terminated at f_h. Practical examples usually require the exact shape of $H_\phi(f)$. Programs exist that numerically compute $\sigma_y(\tau)$ for an arbitrary combination of these 5 noise types [5]. Most sources contain at least three of them plus long-term drift or aging.

2.3 Effects of Frequency Multiplication, Division, and Translation

Frequency multiplication by a factor N is the same as phase amplification by a factor N. For example 2π radians is amplified to $2\pi N$ radians. Since PM noise is the mean squared phase fluctuation, the PM noise must increase by N^2. Thus

$$S_\phi(N\nu_0, f) = N^2 S_\phi(\nu_0, f) + \text{Multiplication PM}, \qquad (13)$$

where Multiplication PM is the noise added by the multiplication process.

We see from Eqs. (8), (9) and (13) that the power in the carrier decreases exponentially as e^{-N^2}. After a sufficiently large multiplication factor N, the carrier power density is less than the PM noise power. This is often referred to as carrier collapse [4]. Ideal frequency translation results in the addition of the PM noise of the two sources [2]. The half power bandwidth of the signal also changes with frequency multiplication.

Frequency division can be considered as frequency multiplication by a factor 1/N. The effect is to reduce the PM noise by a factor $1/N^2$. The only difference is that there can be aliasing of the broadband PM noise at the input to significantly increase the output PM above that calculated for a perfect divider [6]. This effect can be avoided by using narrow band filter at the input or intermediate stages. Ideal frequency multiplication or division does not change $\sigma_y(\tau)$.

Frequency translation has the effect of adding the PM noise of the input signal υ_1 and the reference signal υ_o to that of the PM noise in the nonlinear device providing the translation.

$$S_\phi(\nu_2, f) = S_\phi(\nu_o, f) + S_\phi(\nu_1, f) + \text{Translation PM.} \qquad (14)$$

Thus dividing a high frequency signal, rather than mixing two high frequency signals generally produces a low frequency reference signal with less residual noise.

3. Effect Of Multiplicative Noise

Multiplicative noise is noise modulation power that remains proportional to the signal level. For example consider the case where the gain is modulated by some process with an index β as

$$\text{Gain} = G_o(1+\beta)\cos\Omega\tau \qquad (15)$$

If we assume an input signal given by

$$V_{in} = V_o \cos[2\pi\nu_o t + \phi(t)] \qquad (16)$$

then the output voltage will have the form

$$V_{out} = V_o G_o + V_o G_o \beta \cos\Omega t)\cos[2\pi\nu_0 t + \phi(t)] \qquad (17)$$

The amplitude fluctuation is seen to be proportional to the input signal. Using Eqs. (1) and (7) we can compute the AM noise to be

$$\frac{1}{2}S_a(f) = \frac{\beta^2}{2} \qquad (18)$$

Similarly if the phase is modulated as

$$\phi(t) = \beta\cos[\Omega(t)] \qquad (19)$$

the output voltage will be of the form

$$V_{out} = V_o \cos[\omega\tau + \beta\cos[\Omega(t)] \qquad (20)$$

The phase fluctuation is proportional to the input signal and the PM is calculated using Eqs. (1) and (3) to be

$$\frac{1}{2}S_\phi(f) = \frac{\beta^2}{4} \qquad (21)$$

4. Effect of Additive Noise

The addition of a noise signal $V_n(t)$ to the signal $V_o(t)$ yields a total signal

$$V(t) = V_o(t) + V_n(t) \qquad (22)$$

Since the noise term $V_n(t)$ is uncorrelated with $V_o(t)$, ½ the power contributes to AM noise and ½ the power contributes to PM noise.

$$\text{AM } V_n(t)/\sqrt{2} \quad \text{PM } V_n(t)/\sqrt{2} \qquad (23)$$

$$L(f) = \frac{s_\phi(f)}{2} = \frac{s_a(f)}{2} = \frac{V_n^2(f)}{4V_o^2}\frac{1}{BW} \qquad (24)$$

where BW is the bandwidth in Hz. We see that the AM and PM is proportional to inverse power. These results can be applied to amplifier and detection circuits as follows. The input noise power to the amplifier is given by kTBW. The gain of the amplifier from a matched source into a match load is G_o. The noise power to the load is just $kTBWG_oF$, where F is the noise figure. The output power to the load is P_o. Using Eq. (24) we obtain

$$L(f) = \frac{s_\phi(f)}{2} = \frac{s_a(f)}{2} = \frac{V_n^2(f)}{4V_o^2}\frac{1}{BW} = \frac{2kTBWFG_o}{4P_oBW} = \frac{kTFG_o}{2P_o} = -177 \text{dBc/Hz} \qquad (25)$$

for T= 300K, F=1, , $P_o/G_o = P_{in} = 0$ dBm.

5. Phase Jitter

The phase jitter $\delta\phi$ is computed from the PM noise spectrum using

$$\delta\phi = \int_0^\infty [S_\phi(f)]H(f)df \qquad (26)$$

Generally H(f) must have the shape of the high pass filter or a minimum cutoff frequency f_{min} used to exclude low frequency changes for the integration, or $\delta\phi$ will diverge due to random walk FM, flicker FM, or white FM noise processes. Usually H(f) also has a low pass characteristic at high frequencies to limit the effects of flicker PM and white PM [2]. See Table 1.

6. Timing Jitter

Recall that $\sigma_y(\tau)$ is the fractional frequency stability of adjacent samples each of length τ. See Fig. 1. The time jitter δt is the timing error that accumulates after a period τ. δt is related to $\sigma_y(\tau)$ by

$$\frac{\delta t}{\tau} = \frac{\delta \upsilon}{\upsilon} = \sigma_y(\tau) \qquad \delta t = \tau\sigma_y(\tau) \qquad (27)$$

Table 1 shows the asymptotic forms of $\sigma_y(\tau)$, δt, and $\delta\varphi$ as a function of τ, f_{min}, and f_h for the 5 common noise types at frequency ν_o and $N\nu_o$, under the assumption that $2\pi f_h \tau > 1$. It is interesting to note that for white phase noise, all three measures are dominated by f_h[5]. For random walk frequency modulation (FM) and flicker FM, $\sigma_y(\tau)$ is independent of f_h and instead is dominated by $S_\varphi\ (1/\tau)$ or $S_\varphi\ (f_{min})$. Also, the timing jitter is independent of N as long as we can still identify zero crossings, while the phase jitter, which is proportional to frequency, is multiplied by a factor N. Typical sources usually contain at least 3 of these noise types.

Table 1. $\sigma_y(\tau)$, δt, and $\delta\varphi$ as a function of τ, f_{min}, and f_h at carrier frequency υ_o and $N\upsilon_o$

Noise type	$S_\varphi(f)$	$\sigma_y(\tau)$	δt at υ_o or $N\upsilon_o$	$\delta\varphi$ at υ_o	$\delta\varphi$ at $N\ \upsilon_o$
Random Walk FM	$[v^2/f^4]h_{-2}$	$\pi[(2/3)h_{-2}\tau]^{1/2}$	$T\pi[(2/3)h_{-2}\tau]^{1/2}$	$\upsilon([(1/(3f_{min}^3)h\text{-}_2]^{1/2}$	$N\upsilon([(1/(3f_{min}^3)h_{-2}]^{1/2}$
Flicker FM	$[v^2/f^3]h_{-1}$	$[2\ln(2)h_{-1}]^{1/2}$	$\tau[2\ln(2)h_{-1}]^{1/2}$	$\upsilon([(1/(2f_{min}^2)h_{-1}]^{1/2}$	$N\upsilon([(1/(2f_{min}^2)h_{-1}]^{1/2}$
White FM	$[v^2/f^2]h_0$	$[\{1/(2\tau)\}h_0]^{1/2}$	$[(\tau/2)h_{-0}]^{1/2}$	$\upsilon\{[(1/f_{min})-/f_h)]h_0\}^{1/2}$	$N\upsilon[(1/f_{min})-/f_h)]h_0^{1/2}$
Flicker PM	$[v^2/f^1]h_1$	$[1/(2\pi\tau)]\ [1.038 +3\ln(2\pi f_h\tau)h_1]^{1/2}$	$[1/(2\pi)]\ [1.038 +3\ln(2\pi f_h\tau)h_1]^{1/2}$	$\upsilon[\ln(f_h/f_{min})h_1]^{1/2}$	$N\upsilon[\ln(f_h/f_{min})h_1]^{1/2}$
White PM	$[v^2f^2]h_2$	$[1/(2\pi\tau)][3f_hh_2]^{1/2}$	$[1/(2\pi)][3f_hh_2]^{1/2}$	$\upsilon[f_hh_2]^{1/2}$	$N\upsilon[f_hh_2]^{1/2}$

7. Discussion

We have explored the effects of phase modulation (PM), amplitude modulation (AM), and additive noise on the rf spectrum, phase jitter, timing jitter, and frequency stability of precision sources. Under ideal frequency multiplication or division by a factor N, the PM noise and phase jitter of a source is changed by a factor of N^2. After a sufficiently large N, the carrier power density is less than the PM noise power. This condition is often referred to as carrier collapse. Noise added to a precision signal results in equal amounts of PM and AM noise. The upper and lower PM (or AM) sidebands are exactly equal and 100% correlated, independent of whether the PM (or AM) originates from random or coherent processes.

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