*Short Communication*

# Relativistic red shift with $1 \times 10^{-16}$ uncertainty at the NIST, Boulder*

*M. A. Weiss and N. Ashby*

**Abstract.** The relativistic red shift for the new caesium fountain at the National Institute of Standards and Technology (NIST) at Boulder, USA, has been estimated using two independent methods. Method 1 uses the EGM96 geopotential model to estimate the geopotential number, $C$, and subsequently the frequency offset, $-C/c^2$ ($c$ being the speed of light), obtaining the value $1797.8 \times 10^{-16}$ with an estimated standard uncertainty of $1.1 \times 10^{-16}$. Method 2 uses the geopotential number from the National Geodetic Survey data sheet for the NIST marker and gives a frequency offset of $1798.9 \times 10^{-16}$, with an estimated standard uncertainty of $0.2 \times 10^{-16}$.

## 1. Introduction

With the advent of new primary frequency standards having uncertainties approaching 1 part in $10^{15}$, there is a need for improved estimates of the relativistic red shift. This is a combination of two effects from the theory of relativity. General relativity states that a clock at a higher gravitational potential runs faster. In relativity, "higher" potential means less negative potential. In the relativistic convention, potential generally has negative value, approaching zero as a particle moves towards infinity away from an attracting body. Thus the effect of the geopotential on a clock would cause it to run faster as it moves away from the Earth, or in our case, higher up from the geoid. Note that geodesy uses the *opposite* sign convention for geopotentials; in geodesy, all potentials are positive, so that a higher potential would generally be closer to the Earth. In this paper we use the relativistic convention in which all geopotentials are negative. A second effect in relativity is the so-called second-order Doppler shift, in which a standard clock runs slower as it moves faster, relative to a clock at rest. The rotation of the Earth therefore gives rise to a centripetal potential that also changes the clock's frequency. We differentiate between the potential due to *gravitation* and that due to *gravity*: the former arises from the presence of attracting masses *only*, the latter contains in addition the centripetal potential due to the Earth's rotation. It is the gravity potential that we need to consider here, therefore the term "gravitational red shift" is somewhat misleading and has been avoided.

A primary frequency standard contributing to International Atomic Time (TAI) must be corrected to run at the rate that clocks would run on the Earth's geoid. In order to account for the frequency offset of a standard, the offset in gravity potential (Earth's gravitation plus centripetal potential) from the value on the geoid to the location of the primary frequency standard must therefore be determined. We have used two methods to determine this offset for the NIST at Boulder. First, there is a recent spherical harmonic model of the Earth's gravitational potential field, referred to as EGM96 [1]. This model is complete to degree and order 360. It was developed based on measurements from and to several satellites, and measurements made on and near the Earth. We evaluate this for NIST, Boulder, then add the centripetal potential due to the Earth's rotation to determine the gravity potential. Second, the National Geodetic Survey (NGS) has data collected from spirit-levelling surveys and measurements of gravity over the United States. Data from these campaigns are available and provide the results of discrete line integrals of the acceleration due to gravity (the Earth's gravitation plus centrifugal acceleration) from coastal tide gauges to the location of

M. A. Weiss: National Institute of Standards and Technology (NIST) 847, 325 Broadway, Boulder, CO 80302, USA.
N. Ashby: Department of Physics, Campus Box 390, University of Colorado, Boulder, CO 80309-0390, USA.

markers. In this way they are direct measurements of the change in gravity potential from the reference point (tide gauge) to a marker. There is one such marker at the NIST, and an associated NGS data sheet giving its geopotential number, that is, the change in potential from mean sea level to the location of the marker. To use this, we apply an estimate of the offset of the reference geoid for the NGS system from the geoid used for the EGM96 system.

## 2. Theory

The value of gravity potential on the geoid has an estimated standard uncertainty of 1 $m^2\ s^{-2}$ [2] corresponding to a frequency uncertainty[1] of $0.1 \times 10^{-16}$. However, the physical realization of the geoid surface in an absolute sense, and the determination of its offset from the vertical datum origin, may be known to only 0.20 m. This implies that there is a minimum uncertainty in the standard rate of clocks of $0.2 \times 10^{-16}$ [3, 4]. While we may be able to determine the offset from a particular estimate of the geoid with lower uncertainty than this, such determinations may require future revisions as better estimates are made of the geoid surface.

In this paper we determine the gravity potential offset from a "best estimate" of the geoid, using two methods. In doing so, we also consider to what extent the two reference geoids differ. Our current goal is to determine the relativistic red shift with an uncertainty of no more than $1.0 \times 10^{-16}$ in support of the NIST caesium fountain frequency standard [5]. A change of 1 m in height near the Earth's surface would produce a fractional frequency change in a clock of about $1.1 \times 10^{-16}$. We do not consider lunar and solar tides because their displacement is below the level of 1 m [3].

In comparing spirit-levelling data with the EGM96 model we must be concerned with the inherent uncertainties of both systems, as well as any systematic difference between their reference geoid surfaces. While it is difficult to determine minimum uncertainties, several references indicate uncertainties for both systems below the equivalent of 1 m in orthometric height. For spirit levelling, we are concerned about both the growth of uncertainties in levelling from a tide gauge and the offset from the geoid of the tide gauge's measurement of mean sea level. Comparisons of levelling along independent paths can give a measure of the accumulated uncertainty in levelling. Height differences along the Canada-US border using Canadian levelling data only and American levelling data only show a maximum magnitude of 11 cm [6]. The reference surface for the NGS levelling sites is based on the North American Vertical Datum 1988 (NAVD88). A comparison at sites across North America between NAVD88 orthometric heights and corresponding values obtained from GPS positioning and EGM96-derived geoid undulations has shown a bias of about –30 cm (the NAVD88 reference surface being below the EGM96-implied geoid) [7]. Considering the way in which uncertainties are expected to grow with spirit levelling, and after applying the –30 cm correction, we estimate the uncertainty in orthometric height from NAVD88 data in the NGS data sheets to be 0.20 m, corresponding to a frequency uncertainty for standards of $0.2 \times 10^{-16}$ [6, 8].

The EGM96 model has been tested in a number of ways [1]. We estimate an uncertainty in orthometric height in the area of the NIST, Boulder, of no more than 1 m, owing to the omission and commission errors of EGM96, implying a frequency uncertainty of $1.1 \times 10^{-16}$.

### *2.1 Evaluation of EGM96*

We evaluated the EGM96 gravitational potential model using known methods for evaluating associated Legendre functions [9, 10]. We used $GM_e = 3.986\,004\,418 \times 10^{14}$ $m^3\ s^{-2}$, where $G$ is the universal gravitational constant and $M_e$ is the mass of the Earth (including atmosphere). For the Earth's equatorial radius, we used $a_e = 6378\,136.46$ m, and for its mean angular velocity, $\omega = 7292\,115 \times 10^{-11}$ rad/s.

## 3. Results

We evaluated the EGM96 potential model for the NIST NGS marker, obtaining a gravity potential, $W$ (including the centripetal potential), of $W = -62\,620\,698.8$ $m^2/s^2$. Using the value of the gravity potential on the geoid as $W_0 = -62\,636\,856.88$ $m^2/s^2$, we find the geopotential number, $C = W - W_0 = 16\,158.1$ $m^2/s^2$. Using the speed of light as $c = 299\,792\,458.0$ m/s, we obtain a value for the frequency offset at the marker of $1797.8 \times 10^{-16}$.

An independent measure of the geopotential number is available from the NGS data sheet for the marker on the NIST building. This gives a geopotential number of $C = 16\,170.8$ $m^2/s^2$. Adjusting for the –30 cm offset using the Helmert equation [11], then dividing by the speed of light squared, we obtain a frequency offset of $1798.9 \times 10^{-16}$. Thus we have two values: $1797.8 \times 10^{-16}$ (EGM96 model) and $1798.9 \times 10^{-16}$ (spirit levelling). As these two methods are largely independent, we find consistency in our estimate with an uncertainty of $1 \times 10^{-16}$.

## 4. Future improvements

Based on our current knowledge of the geoid, it seems that existing measurements and models of the Earth's gravitational potential may not support estimates of the relativistic red shift of clocks on the Earth at better than the $10^{-17}$ level. As this value contributes to the error budget of a primary frequency

1. All uncertainties quoted in this paper are standard uncertainties.

standard in a root-mean-square sense, the implication is that a primary frequency standard in an Earth-bound laboratory will have difficulty contributing to TAI at better than the $10^{-16}$ level. In the next decade it seems reasonable that standards may obtain accuracies whose application to TAI are inhibited by this current ability to estimate the relativistic red shift. Indeed such standards may become useful in determining differential gravity potentials across regions of the Earth. One hopeful development that may lead to improvement in estimating the geoid is the Gravity Recovery and Climate Experiment (GRACE) of the National Aeronautics and Space Administration's Jet Propulsion Laboratory. This project's mission states [12]:

"The primary objective of the GRACE mission is to provide gravity models with accuracies that better existing global and high spatial resolution models of the Earth's gravity field by at least an order of magnitude, on a monthly basis, for a period of up to five years."

The five-year period is scheduled for launch in 2001 through to the end of mission in 2006. A 1 cm accuracy in geoid undulation determination is expected. If we could determine the geopotential equivalents at the Earth's surface, we would obtain a relativistic red shift estimate of clock frequency offset at the $10^{-18}$ level.

**Acknowledgements.** We wish to thank Dr Nikolaos Pavlis of Raytheon ITSS Corporation for numerous helpful conversations. We also thank him and Dr Judah Levine of the NIST for their useful comments on this document.

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*Received on 14 February 2000 and in revised form on 10 July 2000.*