

Generalization of the Total variance approach to the modified Allan variance*

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Abstract

The Total variance approach involves periodically extending a data sequence beyond its normal measurement duration and in such a way that a particular time statistic is expected to have the same value with extended data as without. For those statistics which estimate components of broadband noise processes, the approach can significantly reduce the spread or uncertainty in the result. We describe a Total variance approach for improving the confidence of the estimation of the modified-Allan variance (Mvar) for the five common integer power-law noises and which simultaneously has low, easily removable bias. We have found in simulation studies that if a reflection-only extension procedure is applied to Mvar's individual estimates, we obtain a new estimate of Mvar which exhibits an increase in equivalent degrees of freedom at mid- and long-term integration times.

1 Introduction

This writing assumes familiarity with the modified-Allan variance [1] defined as an expectation value of a squared second-difference of averaged time-error measurements, whose maximum-overlap estimator will be called “Mvar” [2]. The modified-Allan variance, like the traditional Allan variance with its maximum-overlap estimator called “Avar,” is suited to processes with stationary second increments. It is designed specifically to extract broadband oscillator and measurement-system power-law noise models with spectral densities following [3]:

$$S_y \sim \text{const} \cdot f^\alpha, \tag{1}$$

where S_y is the spectral density function in terms of fractional-frequency fluctuations $\{y_t\}$ and $-2 \leq \alpha \leq +2$.

Total variance, called “Totvar” (pronounced tōt'-vār) for short, characterizes typical white (WH), flicker (FL), and random-walk (RW) frequency modulation (or f^0, f^{-1}, f^{-2} FM) noise and drift, in addition to white or flicker phase modulation (PM) noise, with better confidence than Avar in terms of equivalent degrees of freedom (edf) [4–6]. However, like Avar, Totvar cannot distinguish white phase modulation (WHPM) noise, $\alpha = +2$, from flicker phase modulation (FLPM) noise, $\alpha = +1$. The response to both is a τ^{-2} slope, thus Totvar (like Avar, see ref. [7]) separates broadband power-law noise into four of the five noise models. Totvar is implemented as follows.

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A whole data run of length T_{max} of time-difference measurements x_t is extended at both the beginning and end by attaching a sign-inverted and mirror-reflection copy of the original data sequence, referred to as an “odd extension” type because of the sign inversion, in effect creating a new virtual sequence $x_t^\#$ of length nearly $3T_{max}$. Avar is computed using the extended virtual sequence and the resulting estimator is denoted Totvar to show that the odd mirror-reflection extension procedure is used. In the presence of FLFM and RWFM, Totvar with respect to Avar has a modest negative bias in long term which can easily be removed. Totvar is a recommended substitute for Avar if long-term measurements of frequency stability are important [8].

Mvar distinguishes WHPM noise, f^{+2} going as τ^{-3} , from FLPM noise, f^{+1} going as τ^{-2} , and thus characterizes the full range of five integer power-law noise types. This property of Mvar makes it ideal for evaluating oscillator instability within synchronization and time-transfer systems, measurement systems dominated by phase-noise, and telecommunications networks [9].

Since the confidence improves for characterizing oscillator noises using Totvar, why not use its routine applied to other time-domain statistics such as, for example, Mvar? The reason is that with WHPM noise, the odd mirror-reflection extension introduces gross bias due to a step at the endpoint match. The step can be eliminated by omitting the sign inversion, using what is called an even mirror-reflection extension. Regrettably, this endpoint match is inappropriate for low-frequency broadband noise such as RWFM. With RWFM noise, short-term τ -values have significant positive bias using the even mirror-reflection extension. The even mirror-reflection extension is nevertheless useful, primarily for characterizing synchronization systems in which RWFM is unlikely to be present [10].

In exploring ways to reduce bias across all five common power-law noise types while still improving confidence in terms of edf, we have found a new statistical routine conceptually based on the Total variance and its data extension idea to obtain a “Total modified Allan variance.” Rather than extending the T_{max} -length data run, creating series $x_t^\#$ and calculating Mvar at varying τ -values on this one sequence, we first remove a background slope associated with the *subsequence* of length $T = 3\tau$ corresponding to *each* subestimate used to compute Mvar at a particular τ -value. This new subsequence, denoted $\{^\circ x_{t,T}\}$, is then extended by an uninverted, or even, mirror-extension, thus creating a new, triple-length subsequence $\{^\circ x_{t,T}^\#\}$ used to compute one “Total Mvar subestimate” at that τ -value. Finally, we average all Total Mvar subestimates obtained from each triple-length subsequence and available from the whole data run to calculate a new Mvar estimator called “mod-Totvar” (at that τ -value). Although a bit more complicated than Totvar, this procedure results in a modestly biased estimate of the modified-Allan variance having significantly increased edf covering five integer power-law noise types, thus the range $-2 \leq \alpha \leq +2$.

This paper reports results using simulation studies. Section 2 evaluates various types of data extensions and establishes the best procedure for a Total approach to Mvar. Section 3 defines the method of computing the estimator mod-Totvar (τ_0, τ) . Section 4 gives the responses, edf values and bias associated with mod-Totvar (τ_0, τ) as compared with the classical Mvar (τ_0, τ) . Section 5 discusses why the name “mod-Totvar (τ_0, τ) ” is suitable for the procedure found by the study presented here.

2 Types of Data Extensions

Four candidate extensions are investigated. An original data sequence $\{x_i\}$ is referred to as a *subsequence* consisting of N_{x_i} measurements, thus $T = (N_{x_i} - 1)\tau_0$. This is actually a piece of a whole data run. It is extended to form a new, larger piece. The four types, each extension method (only the right extension is formulated for simplicity), and period are as follows:

1. An uninverted or even mirror-reflection, $(x_{N+i} = x_{N-i+1})$, $2T$ -periodic.
2. A sign-inverted or odd mirror-reflection, $(x_{N+i} = 2x_N - x_{N-i+1})$, $2T$ -periodic.
3. A straight periodic duplication, $(x_{N+i} = x_i)$, T -periodic.
4. Same as type 3 with end-to-beginning connections, $(x_{N+i} = x_i - x_1 + x_N)$, T -periodic.

Figure 1 illustrates these extension types.

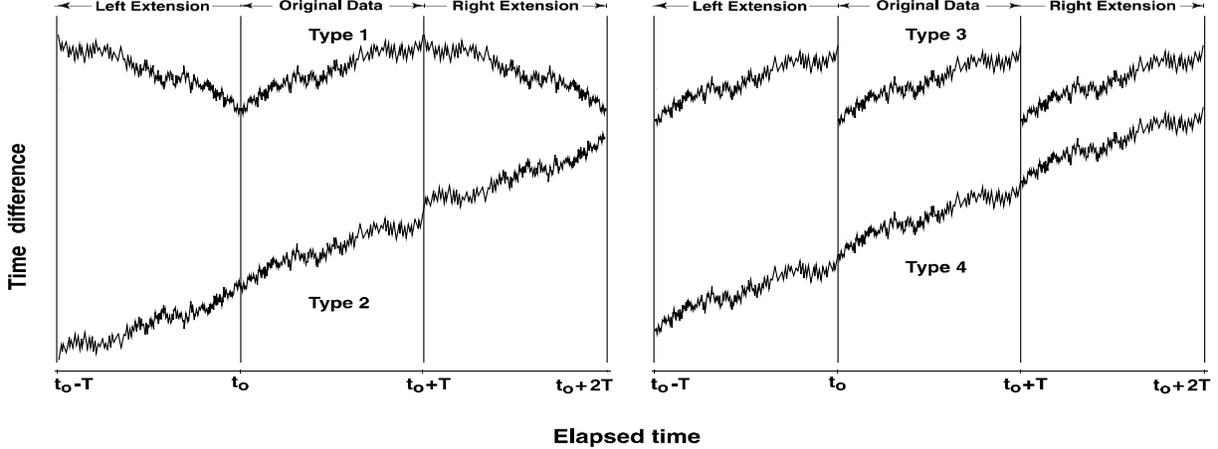


Figure 1: Types of data extensions are: (1) $2T$ -periodic uninverted or even mirror-reflection, (2) $2T$ -periodic sign-inverted or odd mirror-reflection, (3) T -periodic straight duplication, and (4) T -periodic duplication with end-to-beginning connections.

The effect of random endpoint match on Mvar can now be observed by looking at time-shifted calculations of Mvar which go beyond sampling interval T , namely into the extended portion, and in increments of data spacing τ_0 . It is desirable for the average of all time-shifted Mvar values to be least biased for all noise cases relative to the classical Mvar, which is the Mvar value corresponding to a null shift.

Again, the subsequence $\{x_i\}$, whose duration we have stated is a sampling interval T and whose corresponding number of points is N_{x_i} , does *not necessarily* represent the whole data run. For clarity, T_{max} , having $N_{x_{max}}$ total number of points, will designate the duration of the whole data run.

2.1 Reduction of Gross Bias: Selecting the Type of Data Extension

We compute a series of mean values of each subestimate of Mvar and the standard deviation of these subestimates as a function of time-shifting through the extended subsequence. For illustration, let a subsequence consist of $N_x = 769$ points with τ_0 -spacing. For example, if $\tau_0 = 1s$, then $T = 768s$. Furthermore, set integration-time $\tau = 256\tau_0 = 256s$, that is, its max of $\tau/T = 1/3$. 1000 different *i.i.d.* noise realizations of WHPM, FLPM, WHFM, FLFM, and RWFM were generated and followed by the application of each of four extension types in the previous section. For example, Figures 2 to 4 are results of the type 1 extension and show mean values and associated standard deviations (by the error bars) computed at each time-shift of $10\tau_0$, that is, time-shifts of $0s, 10s, 20s, \dots, 750s, 760s, 770s, \dots, 1520s, 1530s$. Each mean value is an average of 1000 estimates, and the first mean value is at a null-shift ($0s$) which corresponds to the classical Mvar result. This value repeats at a time-shift of $768s$ or of $1536s$, which respectively correspond to a T -periodic or $2T$ -periodic extension type and observed to be one full, or circular, period of the mean-value.

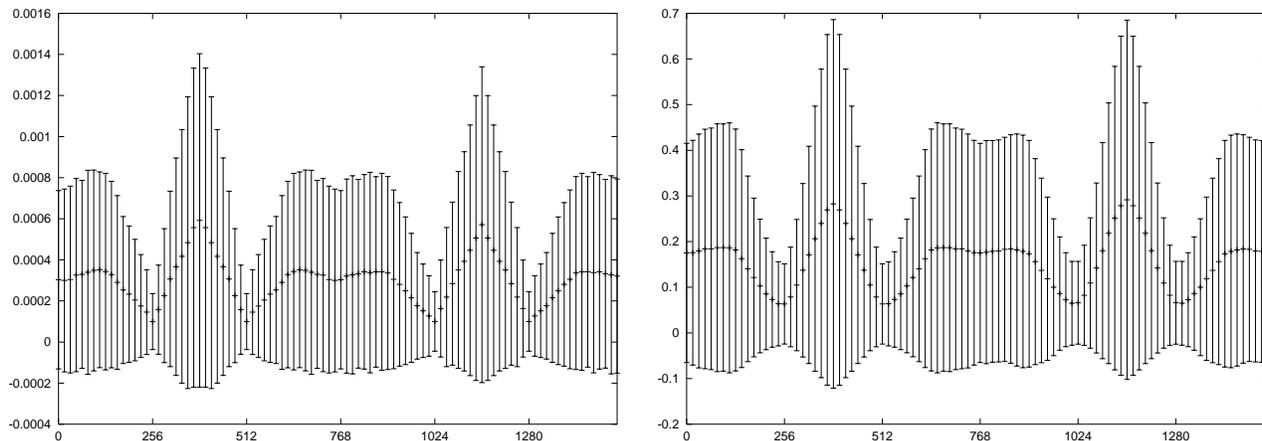


Figure 2: Time-shifted Mvar mean and corresponding 1σ standard deviation of the mean (by the error bars) after extending a simulated subsequence by even reflection (type 1) for subsequence noise types white PM (left) and flicker PM (right).

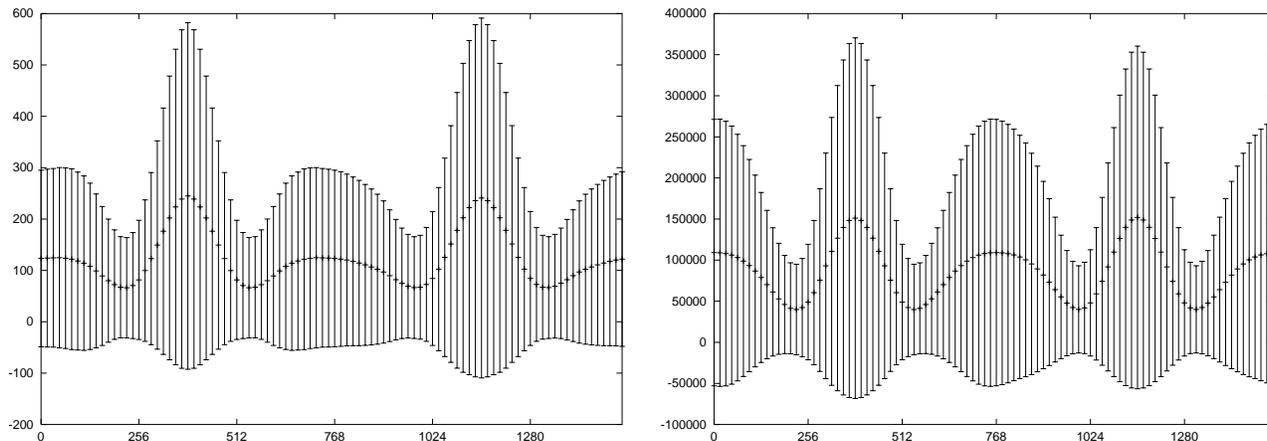


Figure 3: Same as Figure 2 for white FM (left) and flicker FM (right).

The departure of the mean-value of Mvar as a function of time-shift for various noise types is readily visible using this procedure. Our goal is to find a minimum in the difference between the classical, or null-shift, Mvar estimate indicated at time-shift of 0 and an average of all remaining time-shifted subestimates. Each standard deviation, besides showing the uncertainty associated with each time-shifted Mvar value, also serves to indicate the degree of correlation in the data at that value of time-shift as compared to a null shift.

Extension types 2 and 4 introduce a very large bias in the presence of white PM, which is positively peaked at time shifts corresponding to the endpoints. This is because a *phase step* is very likely to occur at the endpoints. This step causes an undesirable positive shift in the mean for the extended segments. Types 2 and 4 however give better results with random-walk FM because the extended sequence is smooth, like the real data.

Extension types 1 and 3 work well with white PM but lack the smoothness typified by random-walk FM. Of the remaining extension types (1 and 3), type 1 (even reflection only, whose results are Figures 2 to 4) has lower departure, hence lower overall bias, as shown in a comparison in

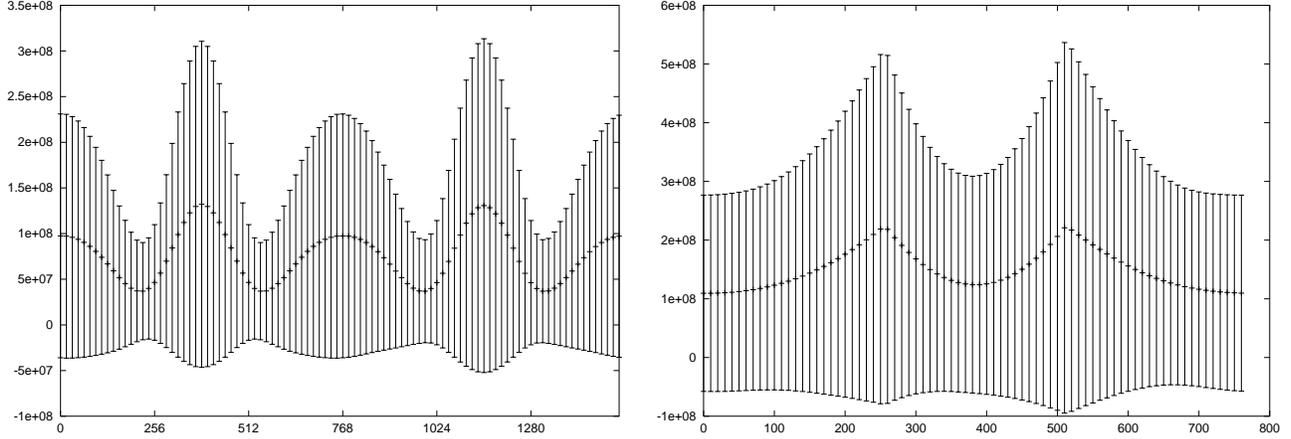


Figure 4: Same as Figure 2 for random-walk FM (left) and using the type 1 extension. For comparison, the right plot shows results using a straight T -periodic extension (type 3) for random-walk FM.

Figure 4 for the specific case of random walk FM noise. Moreover, type 1 involves a triply-extended subsequence, and achieves a greater edf advantage than type 3 which involves only a doubly-extended subsequence. We conclude that a data extension by even reflection (type 1) is the best candidate for constructing Total Mvar.

2.2 The Effect of Frequency Difference and Drift

Each subsequence is likely to be characterized by an offset in frequency as a linear rate offset or linear background in time-deviation data $\{x_i\}$. This causes the evenly reflected subsequence, resulting from a type 1 extension, to have an artificial up-down ramp-function oscillation with period $2T$. This artifact of the reflection-only extension of the subsequence is shown in Figure 1- (Type 1). We can remove a linear fit to each subsequence to remove the linear background and cancel this oscillation.

Removing a linear fit to time-error values in a subsequence is permitted because Mvar is invariant to an overall shift in both phase and frequency. In other words, a first-degree polynomial $c_0 + c_1 i$ which is added to the original subsequence x_i does not change an Mvar result. Thus we are at liberty to arbitrarily *choose* c_0 and c_1 in the subsequence. We will choose them primarily to suppress this spurious spectral component at frequency $1/2T$ arising from reflection-only data extension. Removing a linear fit suitably does this.

As a final note, all of the extension types cause Mvar to be more sensitive to linear frequency drift, a quadratic function in terms of time-deviation. As is the usual practice, an estimate of overall drift should be removed so not to mask the characteristic random noise level.

3 Method of Computation

Given a sequence of time deviates $\{x_n\}, n = 1, \dots, N_{xmax}$, with a sampling period between adjacent observations given by τ_0 , we define the $\tau = m\tau_0$ -average time deviate as

$$\bar{x}_n(m) \equiv \frac{1}{m} \sum_{j=0}^{m-1} x_{n+j}. \quad (2)$$

Let $z_n(m) = \bar{x}_n(m) - 2\bar{x}_{n+m}(m) + \bar{x}_{n+2m}(m)$. By definition

$$\text{mod-}\sigma_y^2(\tau) = \frac{1}{2\tau^2} \langle z_n^2(m) \rangle, \quad (3)$$

where $\langle \cdot \rangle$ denotes an infinite time average over n and $\text{mod-}\sigma_y^2$ actually depends on m , specifically both τ_0 and τ . For simplicity, the τ_0 -dependence of $\text{mod-}\sigma_y^2$ is usually suppressed as in (3). But this τ_0 -dependence is central to the advantage of using $\text{mod-}\sigma_y^2$, and figures prominently as we now construct a Total version. $z_n(m)$ is computed from a data segment or *subsequence* of $\{x_n\}$, consisting of $3m$ points. Define this subsequence $\{\{x_n\}\} = \{x_i\}, i = n, \dots, n + 3m - 1$. Offset the subsequence by removing a linear trend by making

$${}^\circ x_i = x_i - c_1 i,$$

where c_1 is a frequency offset which is removed to minimize $\sum_{i=n}^{n+3m-1} ({}^\circ x_i - \overline{{}^\circ x_i})^2$, to satisfy a least-squared-error criteria for the subsequence. In practice, it is sufficient to remove a background slope computed by averaging the first and last halves of the subsequence divided by half the interval. Now extend the “offset-removed” subsequence $\{{}^\circ x_i\}$ at both ends by an uninverted, even reflection. Utility index l serves to construct the extensions as follows. For $1 \leq l \leq 3m$, let

$${}^\circ x_{n-l}^\# = {}^\circ x_{n+l-1}, \quad {}^\circ x_{n+3m+l-1}^\# = {}^\circ x_{n+3m-l}, \quad (4)$$

to form a new data subsequence denoted as $\{{}^\circ x_i^\#\}$ consisting of the offset-removed data in its center portion, plus the two extensions, and thus having a tripled range of $n - 3m \leq i \leq n + 6m - 1$ with $9m$ points. Now define

$$\text{Totalmod-}\sigma_y^2(\tau_0, \tau) = \frac{1}{2\tau^2} \left\langle \frac{1}{6m} \sum_{i=n-3m}^{n+3m-1} ({}^\circ z_i^\#(m))^2 \right\rangle, \quad (5)$$

where notation ${}^\circ z_i^\#(m)$ means that $z_n(m)$ above is derived from the new triply-extended subsequence $\{{}^\circ x_i^\#\}$. The braces designate that an average is taken over all available n -values (see eqn. (7) below).

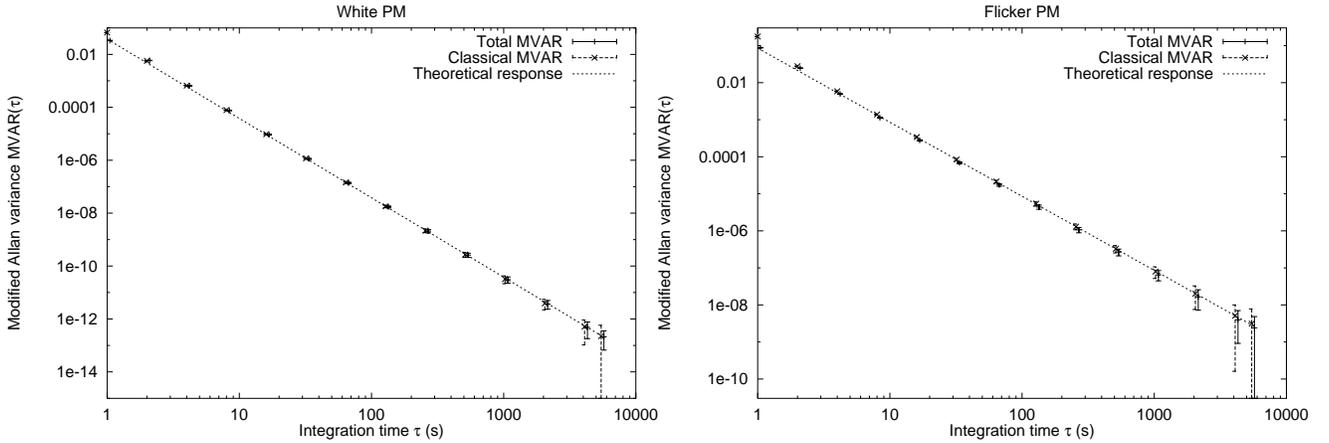


Figure 5: Responses for white PM (left) and for flicker PM (right).

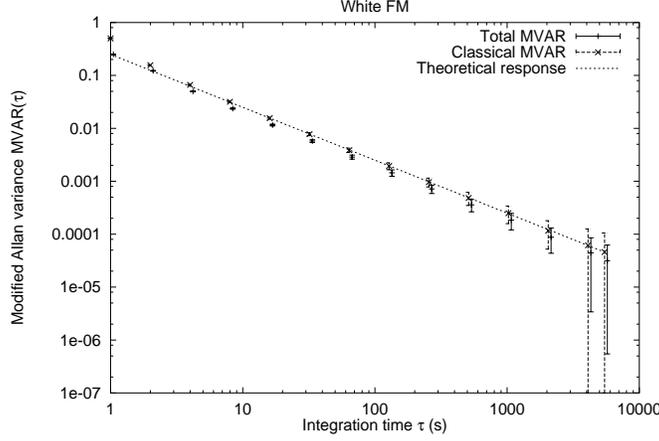


Figure 6: Responses for white FM.

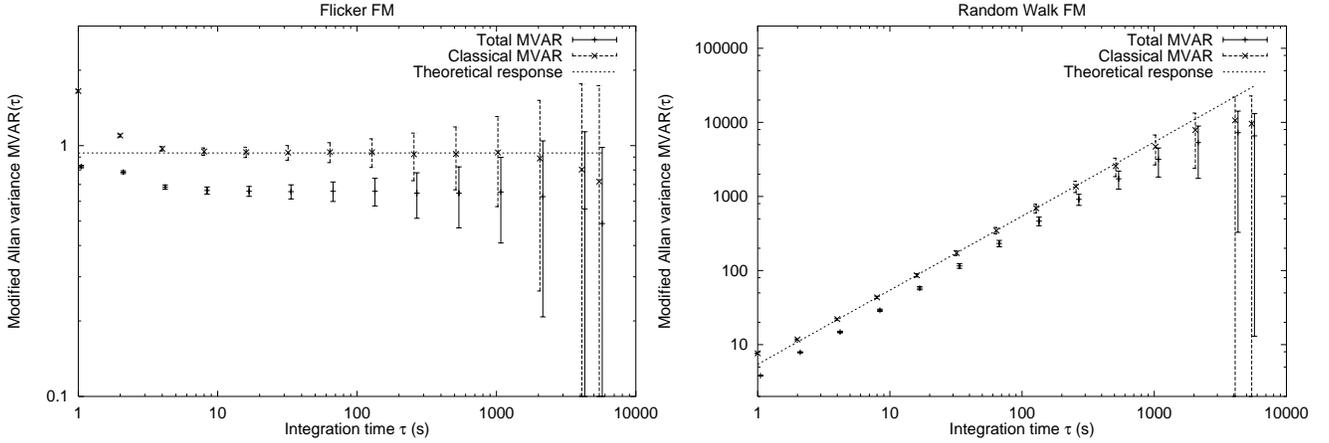


Figure 7: Responses for flicker FM (left) and for random walk FM (right).

A hat “ $\hat{\cdot}$ ” denotes a sample estimate of the function. The maximum-overlap sample estimator, or $\widehat{\text{mod}}-\sigma_y^2(\tau_0, \tau, T_{max})$, is what we have been calling “Mvar” and is given by [1, 2, 9]

$$\widehat{\text{mod}}-\sigma_y^2(\tau_0, \tau, T_{max}) = \text{Mvar}(\tau_0, m, N_{xmax}) = \frac{1}{2(m\tau_0)^2(N_{xmax} - 3m + 1)} \sum_{n=1}^{N_{xmax} - 3m + 1} (z_n(m))^2, \quad (6)$$

for $1 \leq m \leq \lfloor \frac{N_{xmax}}{3} \rfloor$ and $T_{max} = (N_{xmax} - 1)\tau_0$, where $\lfloor c \rfloor$ means the integer part of c . Equation (6) is a simple average of Mvar subestimates given by $\frac{1}{2(m\tau_0)^2} (z_n(m))^2$ in definition (3). “Maximum overlap” means that subestimates (the summand terms in (6)) are overlapping for $m > 1$, and are spaced by τ_0 from which the simple average will have the best confidence in terms of edf [2]. At largest integer $m = \frac{N_{xmax}}{3}$, the summation in (6) consists of only one term, the whole data run “subestimate,” thus representing one degree of freedom.

The corresponding subestimates of $\widehat{\text{Totalmod}}-\sigma_y^2(\tau_0, \tau)$ in definition (5) are given by $\frac{1}{2(m\tau_0)^2} \frac{1}{6m} \sum_{i=n-3m}^{n+3m-1} (z_i^\#(m))^2$. Like Mvar, a maximum-overlap estimator of $\widehat{\text{Totalmod}}-\sigma_y^2(\tau_0, \tau)$

would be a simple average of its subestimates as

$$\widehat{\text{Totalmod}}-\sigma_y^2(\tau_0, \tau, T_{max}) = \frac{1}{2(m\tau_0)^2(N_{xmax}-3m+1)} \sum_{n=1}^{N_{xmax}-3m+1} \left(\frac{1}{6m} \sum_{i=n-3m}^{n+3m-1} (z_i^\#(m))^2 \right). \quad (7)$$

At largest integer $m = \frac{N_{xmax}}{3}$, the outer summation in (7) consists of one term as in (6), but the inner summation is comprised of $6m$ terms. Thus at long-term τ -values corresponding to large values of m , $\widehat{\text{Totalmod}}-\sigma_y^2$ has a sizeable number of estimates which act to reduce the dispersion of variance results. This reduced dispersion is quantified by an increase in equivalent degrees of freedom (see Section 4.2). $\widehat{\text{Totalmod}}-\sigma_y^2$ is called “mod-Totvar” which will be regarded as an improved estimator of the modified Allan variance. Note that its confidence will depend on data spacing τ_0 . This conceptual difference between mod-Totvar and Mvar means that actual measurements should be sampled at a fast rate, at least $m(= \tau/\tau_0) \geq 8$, especially for long-term τ -values, in order to reap the greatest confidence advantage using mod-Totvar.

4 Simulation Study

4.1 Responses to Power-Law Noises

Plots in Figures 5 to 7 compare mod-Totvar, classical Mvar, and a theoretical response for the five power-law noises. Each plot is based on 100 realizations of a particular noise in which $N_{xmax} = 16,384$. The theoretical responses are as follows:

Noise	Theoretical response of mod- σ_y^2
WHPM	$3h_2/(8\pi^2\tau^3)$
FLPM	$(24\ln(2) - 9\ln(3))h_1/(8\pi^2\tau^2)$
WHFM	$h_0/(4\tau)$
FLFM	$(27\ln(3) - 32\ln(2))h_{-1}$
RWFM	$11\pi^2\tau h_{-2}/20$

where h_α is the noise level in terms of the spectral-density of fractional-frequency fluctuations, that is, $S_y(f)$.

4.2 Equivalent Degrees of Freedom

Equivalent degrees of freedom (edf) for statistics such as Mvar depend on m and N_{xmax} . Table 1 compares edf between Mvar and mod-Totvar again from 100 simulation trials in which $N_{xmax} = 16,384$ for each of the five power-law noises. The increased edf using mod-Totvar is significant as m gets large corresponding to long-term τ -values. Plots of Figures 5 to 7 include estimation uncertainties (by error bars) assigned to mod-Totvar using edf values in Table 1 and chi-square distribution properties. These error bars are conservative since the actual distributions are slightly narrower, similar to those found using Totvar at longest integration times [4].

4.3 Bias

From the same set of simulation trials, the bias associated with using mod-Totvar(τ_0, τ) as compared to Mvar is essentially 0 for WHPM, within the uncertainty of the simulation, and progressively goes negative for FLPM, WHFM, FLFM, and RWFM noises. A practical convenience is

Table 1: Comparison of edf’s of Mvar and mod–Totvar and bias of mod–Totdev with 100 simulation trials of $\{x_n\}$ series consisting of $N_{x_{max}} = 16,384$ points each, $\tau = m\tau_0$.

Simulation Results					
$m : \tau$ in units of τ_0	Deg. of Freedom: Mvar mod–Totvar			Bias: $ [1 - \sqrt{\frac{\text{mod–Totvar}}{\text{Mvar}}}] \times 100\%$	
m	WHPM	FLPM	WHFM	FLFM	RWFM
8	3129 4584 -1.6%	1691 2071 -9%	2633 2931 -14%	1709 1843 -16%	1537 1596 -18%
16	1363 2084 -2.2%	1001 1183 -10%	1087 1246 -14%	910 960 -16%	780 810 -18%
32	598 867 -2.3%	502 618 -10%	472 536 -14%	441 472 -16%	366 381 -18%
64	316 458 -2.4%	277 334 -10%	238 265 -14%	240 252 -16%	197 203 -18%
128	178 252 -2.7%	95.2 118 -10%	89.6 101 -14%	117 125 -16%	111 115 -18%
256	70.5 101 -2.5%	62.3 73.2 -9%	55.8 62.6 -14%	43.2 47.1 -16%	66.3 68.6 -18%
512	42.8 62.6 -2.5%	39.3 46.1 -10%	25.7 29 -14%	25.4 26.8 -16%	25.3 26.6 -17.5%
1024	18.1 27.7 -2.2%	16.7 21 -10%	14.9 16.7 -14%	13 14.3 -16%	10.6 11.2 -17%
2048	10.8 15.1 -2.5%	5.1 6.3 -9%	6.6 7.9 -13%	4 4.5 -16%	4.1 4.4 -17%
4096	3.2 5.1 -3.4%	2.1 3.4 -11%	1.8 2.3 -15%	1.4 1.9 -16%	1.8 2.2 -17%
5461	0.8 4.2 -2.8%	0.9 1.9 -12%	1.2 2.1 -17%	1.0 2.0 -17%	1.1 2.0 -17%

that mod–Totvar’s bias is modest and uniformly distributed across all τ -values. Table 1 gives the resulting set of percentage errors in terms of usually-reported deviations, that is, percentage error between mod–Totdev and classical Mdev.

5 Suggested Name: mod–Totvar (τ_0, τ)

The terminology “modified” Allan variance, with estimator Mvar, has been used to distinguish its function, namely extracting estimates of the levels of five power-law noises, in contrast to the standard Allan variance, which separates four of the five. To minimize confusion and be consistent with existing terminology, we will refer to the variance of this paper as the “modified Total variance”, shortened to “mod-Totvar” to distinguish it from the standard Total variance. Terminology such as “Total Mvar” is also appropriate and would not be confusing, but suggests to someone getting acquainted with the total concept that the same procedure for Total variance can be used equally for mod-Allan variance, which is not true. Classical Avar and classical Mvar have very different statistical properties. Since modification to the standard Total variance routine is considerable, the authors suggest the use of the name “modified Total variance”. Hence, the usually-reported square-root of such a plot would be called “modified Total deviation.”

6 Conclusion

Total estimators have been based upon the hypothesis that for segment $\{x(t) : t_0 \leq t \leq T\}$, reasonable extensions for $t < t_0$ and $t > T$, can be formed by tacking on reversed versions of $\{x(t)\}$ at the beginning and end of this part of the function [11]. We have applied this approach to the modified-Allan variance, resulting in improved confidence at mid- and long-term integration times. Bias is small and easily removed.

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