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# Total Variance Explained*

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### 1. Abstract

I explain the difference between the Total variance and the Allan variance and what is gained for estimating frequency stability especially at long term. I also describe the property that adding up Total variance values in the usual "power-of-2" increments yields twice the sample standard variance.

### 2. Introduction and Summary

Total variance uses all of the available time-difference data $\{x_n\}$ like the standard variance in a data run of duration $T$ (or length $N_x$ of sample $x_n$-values), but unlike the Allan variance which uses only three (first, median, and last values). I describe a model of the Allan variance as a measure of asymmetry of first and last points with respect to the median by the misalignment of this triplet over $T$. The concept of Total variance involves scanning the entire interval for all such asymmetries about the median value and averaging them. We still obtain a convergent $\tau$-domain variance for all typical oscillator FM power-law noise types (white denoted as WHFM, flicker as FLFM, and random-walk as RWFM). Remarkably, however, the equivalent degrees of freedom (or edf) shows an increase from 1 to 1.5, 2.1, and 3 in the presence of RWFM, FLFM, and WHFM respectively for computations at $\tau = T/2$ thus supporting the basic model criteria. Because of its efficient use of available data, Total variance is recommended for long-term measurements of frequency stability [1]. A particularly simple and recommended computation of frequency stability dubbed "Totvar" has demonstrated improved estimation of frequency stability at long-term $\tau$-values while essentially computing the usual max-overlap sample Allan variance at short- and mid-term $\tau$-values.

### 3. Measuring Frequency Stability

If the time or the time fluctuations between two oscillators can be measured directly, an advantage is obtained over just measuring frequency fluctuations. The reason is that we can readily see time behavior from actual measurements, and frequency can easily be inferred from time. To avoid measurement-system dead time and simultaneously measure the underlying frequency stability of the best oscillators often nearly at the same frequency, we use the dual mixer time difference (DTMD) scheme [2]. Measurement samples of time fluctuations occur at a rate $f_s$ having an interval $\tau_0 = \frac{1}{f_s}$. Given a sequence of time deviates $\{x_n : n = 1, \ldots, N_x\}$ with a sampling period between adjacent observations given by $\tau_0$, we define the $m\tau_0$-average fractional frequency deviate as

$$\overline{y}_n(m) \equiv \tfrac{1}{m} \textstyle\sum_{j=0}^{m-1} y_{n-j},$$

where $y_n = \frac{1}{\tau_0}(x_n - x_{n-1})$ and we regard $\{\overline{y}_n(m) : n = m, \ldots, N_y\}$ as a finite realization of a stochastic process $\{\overline{Y}_n(m) : n = 0, \pm 1, \pm 2, \ldots\}$. Allan in ref. [3] devised a characterization of frequency stability based on an ensemble average of a 2-sample standard deviation. The Allan variance is defined as [4]

$$\sigma_y^2(m) \equiv \frac{1}{2} E\left\{\left[\overline{y}_{n+m}(m) - \overline{y}_n(m)\right]^2\right\},$$

$$\text{and } E\left\{\left[\overline{y}_{n+m}(m) - \overline{y}_n(m)\right]\right\} = \mathrm{Dr}(m),$$

where $E$ throughout this paper means an expected average or infinite mean, and $\mathrm{Dr}(m)$ is a linear trend and assumed to be linear frequency drift which is usually estimated and removed. If the first difference $\{\overline{y}_n(1) - \overline{y}_{n-1}(1)\}$ is stationary, then the stochastic process is such that the expectations above depend on the averaging time index $m$ but not on the time index $n$. Note that each point estimate of the Allan variance computed at $m$ requires a $2m$ interval. A hat " $\hat{}$ " denotes a sample estimate of the function.

The usual max-overlap sample Allan variance $\hat{\sigma}_y^2(\tau, T)$ involves averaging time $\tau = m\tau_0$ and sample data run $T$. Called Avar, it is given by [2, 5]

$$\hat{\sigma}_y^2(\tau, T) = \mathrm{Avar}\left(m, \tau_0, N_{x,y}\right) =$$

$$\tfrac{1}{2(N_y-(2m-1))} \textstyle\sum_{n=m}^{N_y-m} \left(\overline{y}_{n+m}(m) - \overline{y}_n(m)\right)^2 = \quad (1)$$

$$\frac{1}{2(m\tau_0)^2(N_x-2m)} \sum_{n=m+1}^{N_x-m} \left(x_{n+m} - 2x_n + x_{n-m}\right)^2, \quad (2)$$

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for $1 \le m \le \frac{N_x-1}{2}$. Known as a central difference, summand terms in (2) involve a second difference of $\{x_n\}$ expressed symmetrically over a $2m\tau_0$ span. Each central difference consists of only a first, middle, and last $x_n$ value, or $x_n$'s taken in triplet, then subsequently squared and averaged. Having removed drift and other deterministic error sources, an oscillator's random FM noise will have triplets which *on average* fall on a straight line. Thus, any particular triplet is more likely to fall nearly on a straight line, hence the central difference is often uncharacteristically low as judged by the rest of the $x_n$ values in the $2m\tau_0$ interval.

## 4. Motivating Concept of Total Variance

Total variance [6–10], denoted as $\sigma^2_{total}(\tau, T)$, has been developed to exploit the time fluctuations in and around Avar's forementioned triplets on the belief that we are at liberty to choose a range of neighboring quantities which can serve as proper surrogates and average them to obtain a better estimator $\hat{\sigma}^2_y(\tau, T)$. These "proper surrogates" originate in the fact that Avar measures only a *symmetry*, or a lack thereof, in equispaced triplet values of $x_{n\pm m}$. Total applies a *consistency hypothesis* in addition to measures of symmetry. "Consistency" means that *averages* of certain individual estimates can equally serve as any other individual estimate. Since oscillator and clock designers seek persistent frequency, it follows that estimates of frequency and frequency stability can be derived in a variety of ways which take advantage of consistency within statistical variability.

For example, the form of Avar in (1) originated because values of $\{\overline{y}_n(m)\}$ are actually measured asymetrically with respect to $\{x_n\}$ values, that is, *post facto* in which $\overline{y}_n(m) = \frac{1}{m\tau_0}(x_n - x_{n-m})$. For symmetry however, we can substitute

$$\overline{y}^\circ_n(m) = \frac{1}{m\tau_0}\left(x_{n+\frac{m}{2}} - x_{n-\frac{m}{2}}\right). \qquad (3)$$

By a linear interpolation, $\overline{y}^\circ_n(m) =$ average of:

$$\frac{1}{m\tau_0}\left(x_{n-m} - x_n\right) \text{ and } \frac{1}{m\tau_0}\left(x_n - x_{n+m}\right),$$

whose result is $\frac{1}{2m\tau_0}\left(x_{n+m} - x_{n-m}\right)$, or equivalently $\overline{y}^\circ_n(2m)$. In other words, we can use $\overline{y}^\circ_n(m)$ defined symmetrically in (3) in place of $\overline{y}^\circ_n(2m)$. $\overline{y}^\circ_n(2m)$ can in turn be used as a "surrogate value" of $\overline{y}_n(2m)$ in addition to the usual asymmetrically-computed values of $\overline{y}_n(2m)$ in (1) or (2) for the Allan variance estimate $\hat{\sigma}^2_y(2m)$. Likewise, $\overline{y}^\circ_n(2m)$ can serve in place of $\overline{y}_n(4m)$. We can extend this idea to computations at $\overline{y}^\circ_n(4m)$, $\overline{y}^\circ_n(8m)$, and so forth, and find by this pyramid method a greater number of terms that can serve as quantities in estimation (1) while still maintaining its basic properties. Repeating this process reveals a pattern which, when combined with either (1) or (2), serves as a motivating idea for a new statistic called "Total variance."

## 5. Implementing the Concept

Using a symmetry argument, we can define a useful expectation which is independent of values of averaging time index $m$ and the time index $n$. Redefine $\overline{y}_n(m)$ as centered at $n$ by

$$\overline{y}_n(m) \equiv \frac{1}{m}\left(\sum_{j=0}^{\frac{m}{2}-1} y_{n-j} + \sum_{j=0}^{\frac{m}{2}-1} y_{n+j}\right), \qquad (4)$$

or in terms of $\{x_n\}$ values, $\overline{y}_n(m) = \frac{1}{m\tau_0}(x_{n-\frac{m}{2}} - x_{n+\frac{m}{2}})$, for $m$-even. Random FM noise processes also center before and after $n$ such that,

$$E\left\{\overline{y}_{n+\frac{m}{2}}(m) - \overline{y}_n(m)\right\} = E\left\{\overline{y}_{n-\frac{m}{2}}(m) - \overline{y}_n(m)\right\}.$$

Therefore

$$E\left\{\overline{y}_{n+\frac{m}{2}}(m) - \overline{y}_{n-\frac{m}{2}}(m)\right\} = 0,$$

and we can derive the symmetric form of the Allan variance as

$$\sigma^2_y(m) \equiv \frac{1}{2}E\left\{\left[\overline{y}_{n+\frac{m}{2}}(m) - \overline{y}_{n-\frac{m}{2}}(m)\right]^2\right\}. \qquad (5)$$

Substitute $s$ for $m$ in (4), and define (5) in terms of $\overline{y}_n(s)$ to obtain

$$\sigma^2_{y^\circ}(m,s) \equiv \frac{1}{2}E\left\{\left[\overline{y}_{n+\frac{m}{2}}(s) - \overline{y}_{n-\frac{m}{2}}(s)\right]^2\right\}. \qquad (6)$$

The separation between any two $\overline{y}$ samples in (6) is still $m$ as in (5), but now we are in a position to positively and negatively vary or symmetrically "modulate" the averaging-time $s$ of each $\overline{y}$ in the neighborhood of its usual value $m$ by a small range, say, $\pm\delta$. This has the effect of smoothing $\sigma^2_y(m)$ by using interpolated or "surrogate" values in addition to the usual values of $\overline{y}_{n+\frac{m}{2}}(m) - \overline{y}_{n-\frac{m}{2}}(m)$ as described in the previous section. Finally, we increment $n$, repeat the process, square differences and average to obtain the value of $\hat{\sigma}^2_y(m)$. Generally speaking,

$$\sigma^2_{total}(m) = \overline{\sigma^2_{y^\circ}(m,s)} = \text{smoothed version of } \sigma^2_y(m).$$

The smoothing operation above picks up additional estimates of $\sigma^2_y(m)$ as $m$ increases, improving the usual max-overlap Allan variance estimate, especially if that estimate is uncharacteristically high or low. Equation (6) and hence $\sigma^2_{total}(m)$ differ from the Allan variance because each term constituting a $2m$ interval yields its result dependent on $m$ <u>and</u> $s$. This is because the difference-pair of average frequencies $\overline{y}_{n+\frac{m}{2}}(s)-\overline{y}_{n-\frac{m}{2}}(s)$ may be separated when $s < m$ or overlapped when $s > m$. They are conjoined, or adjacent only when $\delta = 0$ making $s = m$. With WHFM, $\sigma^2_{total}(m)$ is unbiased with respect to $\sigma^2_y(m)$ and is biased negatively with FLFM and RWFM. The bias depends on the depth of modulation $\delta$ relative to $m$ but ought to be limited to $\delta = \frac{m}{2}$ or a full range given by no more than $m$ itself. This is so that $\sigma^2_{total}(m)$ is controlled in a reasonable manner over data run of length $N_x$, an issue discussed next.

Returning to measuring frequency stability from an actual data run, recall that $m$ is a parameter which defines interval $\tau = m\tau_0$. The estimate of $\sigma^2_{total}(\tau)$ for a data duration $T$ involves two issues. The first is that our ability to smooth becomes more and more restrictive as $m \to N_x$ (or equivalently, as $\tau \to T$) because the extent with which we can modulate $m$ and increment $n$ is bounded by the beginning and end points of a fixed-length data run. Second, I have mentioned one way to more fully use the available data. There are other ways from which *ad hoc* manipulation and averaging of $\overline{y}_{n\pm\frac{m}{2}}(s)$ differences can yield improved estimates of the Allan variance. In concept, any could be designated as a "Total" variance if it includes essentially all average frequency differences in a symmetric *totality* over interval $2\tau$ (to be consistent with the Allan variance). One way in particular is simple to implement, has been tested on the expected FM noises and other oscillator error sources, and has an important connection to the classical standard variance explained in the last section. This Total variance evolved from experiments in which the usual max-overlap Avar estimator was applied to periodic extensions of the original data [6].

The most efficient analysis of frequency stability would be to apply forementioned surrogate values where they are needed most, namely at long-term $\tau$-values, beyond $\tau = m\tau_0 = 10\%$ of the data run $T = N_x\tau_0$. For a given $\tau$-value and data run of length $T$, the number of samples in the standard estimator for the Allan variance is of order $\frac{T}{\tau}$, thus (2) is usually more than adequate for determining noise level at short- to medium-term $\tau$-values. A recommended characterization of frequency stability now includes the use of Total variance for the range at long term $\frac{\tau}{T} \geq 10\%$ [1]. This recommendation resulted from what can be regarded as a "hybrid estimator" which uses the standard Allan estimator for short- and medium-term $\tau$-values and rather conveniently applies more surrogate values in long term until they are all applied at the usual Avar function limit of $\tau = \frac{N_x-1}{2}\cdot\tau_0$, half the data run. Such an estimator, now dubbed "Totvar," was introduced in [7]

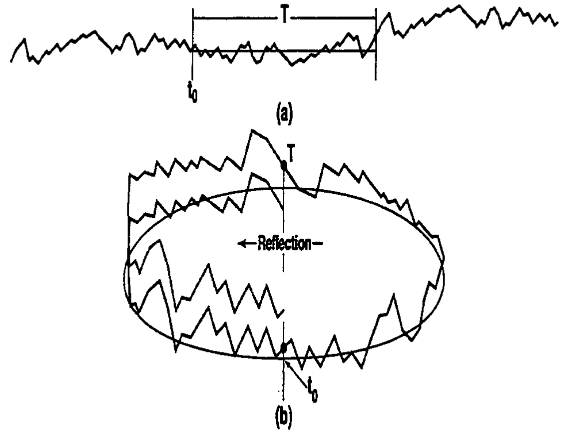


Figure 1: Circular extension of the original $x(t)$ data set for computation of Totvar. (a) Extension of a phase record by "reflection" at both ends; (b) circular representation of extended phase record.

and refined in [9] and [10]. Totvar here is regarded as a composite function of the normal Avar statistic $\hat{\sigma}^2_y(\tau,T)$ and a sample Total statistic $\hat{\sigma}^2_{total}(\tau,T)$ as

$$\text{Totvar}(\tau,T) = f(\hat{\sigma}^2_y(\tau,T), \hat{\sigma}^2_{total}(\tau,T)).$$

Thus Totvar is always a sample variance of a data run of length $T$. The definition of Totvar is

$$\text{Totvar}(\tau,T) = \text{Totvar}(m,\tau_0,N_x) =$$

$$\frac{1}{2(m\tau_0)^2(N_x-2)}\sum_{n=2}^{N_x-1}\left(x^{\#}_{n-m} - 2x^{\#}_n + x^{\#}_{n+m}\right)^2, \quad (7)$$

for $1 \leq m \leq N_x - 1$ where an extended virtual sequence $\{x^{\#}_n\}$ is derived as follows: for $n = 1$ to $N_x$ let $x^{\#}_n = x_n$; for $j = 1$ to $N_x - 2$ let

$$x^{\#}_{1-j} = 2x_1 - x_{1+j}, \quad x^{\#}_{N_x+j} = 2x_{N_x} - x_{N_x-j}. \quad (8)$$

Constructing the extended virtual sequence as in (8) is illustrated in Figure 1 and is called *extension by reflection*. Because of the symmetry of the extended data, the number of summands in (7) does not depend on $m$. Surrogates values in Totvar emerge from this data extension. Rather than doing extensions of the original vector $\{x_n\}$ and applying the straight second-difference, we alternatively can resample within the original vector. Applied to Totvar, this exercise only points out that the procedure used in the sampling function is intricate and not very intuitive because sampling on $\{x_n\}$ is no longer in terms of equispaced triplets spaced $2\tau$ [9]. For example, note that $\tau$ can go to $(N_x-1)\tau_0$ in (7)-(8) instead of the usual limit of $\lfloor(N_x-1)/2\rfloor\tau_0$.

Totvar can also be defined in terms of extended normalized frequency averages by

$$\text{Totvar}(\tau, T) = \text{Totvar}(m, \tau_0, N_y + 1) =$$

$$\frac{1}{2(N_y-1)} \sum_{n=2}^{N_y} \left[\bar{y}_n^{\#}(m) - \bar{y}_{n-m}^{\#}(m)\right]^2, \qquad (9)$$

where $\bar{y}_n^{\#}(m) = \left(x_{n+m}^{\#} - x_n^{\#}\right) / (m\tau_0)$.

The concept of Total variance was motivated by a simple need for improved long-term estimation of the Allan variance and encouraged by the results using a straight circularization technique on original data $\{x_n\}$, but this technique could not work in the presence of RWFM and/or significant drift [6,11]. Totvar here is classed as a hybrid statistic combining the benefits of sample Allan variance with sample Total variance. Totvar has been tested on the range of FM power-law noise types [4] and other expected oscillator and measurement-system error sources [2,7].

We can illustrate (7)-(8) as a hybrid statistic in the following way. If $m = 1$, the virtual extension $\{x_n^{\#}\}$ only needs to be $\tau_0$ longer than $\{x_n\}$ at both ends to compute (7). Thus (7) is essentially the standard Allan estimator (2). As $m$ increases, the virtual extension needs to be longer until at $m = \frac{N_x-1}{2}$, the extensions at each end are length $\frac{N_x-1}{2}$. Of course, there is no standard Allan estimator in the region $\tau > \frac{T}{2}$, so if the "hybrid" called Totvar in (7) is allowed to compute values for $m > \frac{N_x-1}{2}$, it reverts to a region defined by the Total variance but not the Allan variance. Computations of Totvar should not extend beyond $\tau = \frac{T}{2}$ to be consistent with the limit of the standard Allan estimator, but these higher order terms will be considered in the last section.

For accurately estimating the Allan variance, an adjustment must be made to the hybrid estimator Totvar as defined by (7) and its extension $\{x_n^{\#}\}$ in (8) to remove a normalized bias (denoted as nbias) which depends on the ratio $\frac{\tau}{T}$ and whether the noise type in long-term is FLFM or RWFM rather than WHFM. The most noteable adjustments using Totvar in this manner involve formulae for nbias and increased edf compared to the Allan estimator. These can be summarized as [10]

$$\text{nbias}(\tau) = \frac{E\{\text{Totvar}(\tau, T)\}}{\sigma_y^2(\tau)} - 1 = -a\frac{\tau}{T}, \qquad (10)$$

$$\text{edf}(\tau) = \text{edf}[\text{Totvar}(\tau, T)] \approx b\frac{T}{\tau} - c, \qquad (11)$$

where $0 < \tau \leq \frac{T}{2}$ and $a$, $b$, and $c$ are given in Table 1. The values of nbias and edf for the important longest-term case $\tau = T/2$ are tabulated in Table 2. The edf formula (11) is empirical, with an observed error below 1.2% of numerically computed exact values; the tabulated values of edf $(T/2)$ in Table 2 are exact.

Table 1: Coefficients for computing normalized bias and edf of Totvar in the presence of FM noises.

| Noise | $a$ | $b$ | $c$ |
|---|---|---|---|
| WHFM | 0 | $3/2$ | 0.000 |
| FLFM | $(3\ln 2)^{-1}$ | $24(\ln 2)^2 \pi^{-2}$ | 0.222 |
| RWFM | $3/4$ | $140/151$ | 0.358 |

Table 2: Tabulated exact quantities for $\tau = T/2$.

| Noise | nbias$(T/2)$ | edf$(T/2)$ |
|---|---|---|
| WHFM | 0 | 3.000 |
| FLFM | −0.240 | 2.097 |
| RWFM | −3/8 | 1.514 |

Both Totvar and Avar are invariant to certain manipulations of the vector $\{x_n\}$. The simplest example is that we can reverse and/or invert the sequence $\{x_n\}$ without affecting either's result. Unlike Avar's simple sampling function however, Totvar's many sampling functions between $\tau_0 \leq \tau \leq \frac{T}{2}$ are complicated and can be derived from formulae in ref. [9], but unraveling useful information from them is difficult. It is as informative and easier to look at the frequency-response function associated with Totvar compared to Avar as in Figure 2 for a comparison of the effect of their sampling functions. The dashed curve in Figure 2 is a constant-$Q$, one-octave passband filter response considered to be ideal for extracting typical power-law noise levels [12–16]. Totvar implements a circular convolution of Avar's frequency response, thus significantly reducing the depth of periodic nulls.

### 6. Uncertainty of Estimates

Returning to the topic of characterizing noise, the reason for using Totvar is for very efficient extraction of commonly-encountered integer power-law noise types and levels of an oscillator's spectral FM noise. This means greater certainty in the extraction of these parameters and others such as drift and quasi-sinusoidal modulation shown in Figure 3. In retrospect, the formulation of the two-sample Allan ensemble average will contain $N_{\overline{y}(m)}$ average frequencies and only $\frac{N_{\overline{y}(m)}}{m} - 1$ independent intervals with which to do a computation. This represents the actual number of "degrees of freedom". In general the $1\sigma$ accuracy of the computation for $N_{\overline{y}(m)}$ sets is simply given by

$$\%\ error,\ Allan\ deviation = \frac{100}{\sqrt{2(\frac{N_{\overline{y}(m)}}{m} - 1)}}. \qquad (12)$$

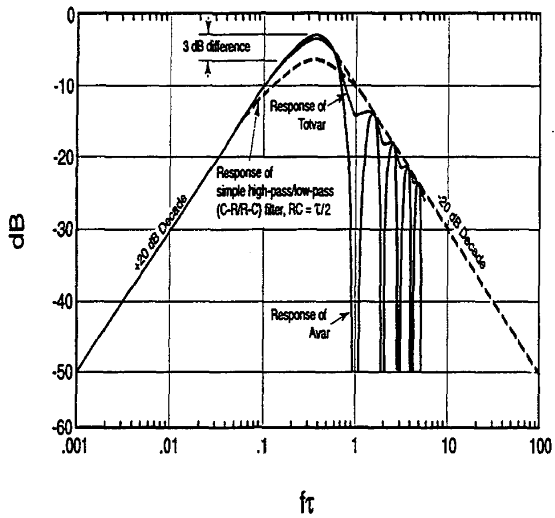


Figure 2: A comparison of frequency responses of Avar (solid curve), Totvar (shaded curve), and a passband variance consisting of a simple cascade of a single-pole high-pass followed by a low-pass filter with identical break points at $RC = \tau/2$ (dashed curve [13]).

This expression is an uncertainty which is adequate for a quick upper-bound approximation for a confidence interval or error bars above and below each value of the Allan deviation *vs.* $\tau$. We assume that the probability distribution is chi-squared, and exact confidence intervals can be determined based on the equivalent degrees of freedom (edf) in overlapping statistical averages for a given noise type, rather than the actual number [5, 17].

Sample estimates of the Total variance (given by the Total deviation or root Totvar, convenient plots of interest) have edf's that are greater than even those using the max-overlap estimates of the Allan variance, and significantly greater at long-term $\tau$-values. Chi-squared distribution functions are used for the Allan variance, but it turns out that the distribution functions are slightly narrower using Totvar (which is another of its benefits) at long averaging times. Thus a conservative upper-bound approximation for either the Total deviation or root Totvar $1\sigma$ accuracy is

$$\%\ error,\ Total\ deviation = \frac{100}{\sqrt{2(\mathrm{edf})}}, \qquad (13)$$

where edf values are conveniently derived from (11) and Table 1 based on the computation of the Total deviation as a ratio of $\tau$ to the length of the data run $T$, rather than the Allan deviations's $\frac{N_{\overline{y}(m)}}{m}$ independent sets as above.

## 7. Analysis of Variance

Consider a function of independent variables. In analysis of variance, we explain the total variability of the function in terms of each variable. In the discussion here, we address functionals which depend on a time interval $\Delta t$. At this point we can recall a conservation principle regarding the standard sample variance, which states that the mean of the interval variances plus the variance of the interval means equals the standard variance of the entire series. This is true for any process, stationary or not. An analysis of variance in terms of the mean of $k$ interval variances and variance of the $k$ interval means is derived in Appendix I.

The standard variance of finite series $\{X_{ij}\}$ in Appendix I is simply a number, partly due to the variance of $k$ interval means and the remaining part due to the mean of $k$ interval variances. Now consider intervals of duration $\Delta t$ and a whole data run of length $T$. The longest possible set of equal-length intervals would be $\Delta t = T/2$, thus there are $k = 2$ consecutive interval means. We recognize that the variance of such two-interval means is the special-case two-sample variance equaling $\frac{1}{2}\hat{\sigma}_y^2(T/2)$, half the sample Allan variance at $\tau = T/2$. But half the sample Allan variance will differ from the standard variance by a remaining portion attributable to the sample variance *within* each of the two intervals by the conservation principle just stated. By double-sampling at $\Delta t = T/4$, we find the two-sample variance ($k = 2$) now must consist of two *non*overlapped variance estimates whose average, denoted as $\hat{\sigma}^2_{y,\text{nono}}(T/4)$, would be the remaining portion if that were as far as the data could be sampled. Repeating this process until there are no remaining interval variances left unaccounted for, we find that

$$\tfrac{1}{2}\left(\hat{\sigma}^2_{y,\text{nono}}(\tau_0) + \hat{\sigma}^2_{y,\text{nono}}(2\tau_0) + \ldots \right. \qquad (14)$$
$$\left. + \hat{\sigma}^2_{y,\text{nono}}(T/4) + \hat{\sigma}^2_{y,\text{nono}}(T/2)\right) = \hat{\sigma}^2_{std}(T),$$

where $T = m\tau_0, m = 2^j$ for $j = 0, 1, \ldots, J-1$ and the nonoverlapped estimator of the Allan variance is

$$\hat{\sigma}^2_{y,\text{nono}}(2^j) \equiv \frac{2^j}{2N_y} \sum_{k=1}^{\frac{N_y}{2^{j+1}}} \left[\overline{y}_{2k2^j}(2^j) - \overline{y}_{(2k-1)2^j}(2^j)\right]^2.$$

The composite in (14) is a common property of what is called a "multiresolution pyramid" [18]. The nonoverlapped $k = 2$ condition requires that the $\tau$-intervals occur in power-of-two increments. This nonoverlapping sample Allan variance would relate

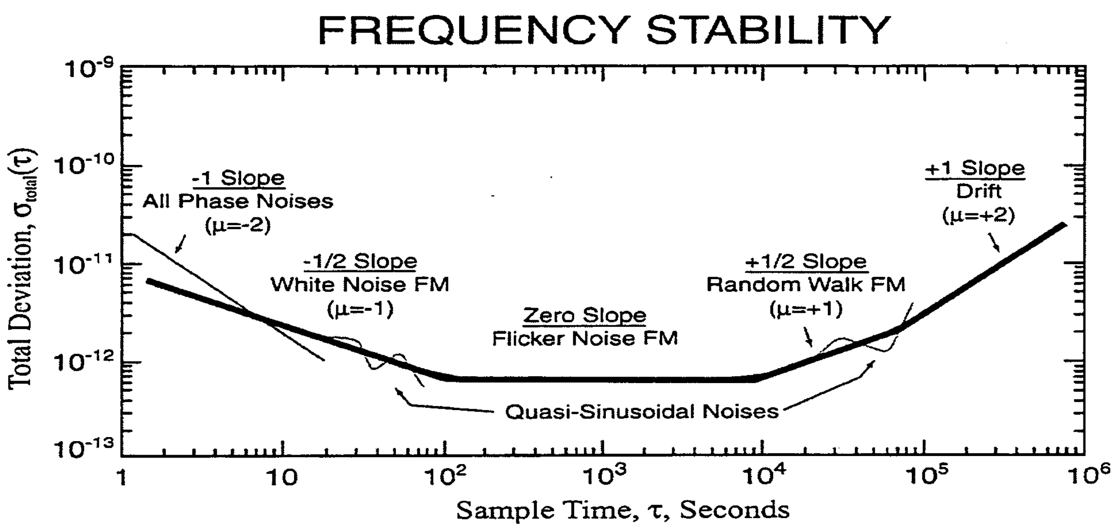


Figure 3: Total deviation plot (or root Totvar) showing power-law noises as straight lines in addition to other error sources. The goal is to identify noise sources and accurately estimate their levels with this kind of frequency stability plot.

directly to the sample standard variance as in (14) but is inefficient as an estimator [15]. Unfortunately the sample Allan variance from its *definition*, for example at $\Delta t = T/4$, calls for *three* variance estimates, not two nonoverlapped, because its definition includes a $\tau = \Delta t$ overlap and the straightforward relationship to the standard variance is lost rather quickly. In otherwords, even for a short series,

$$\frac{1}{2}\left(\hat{\sigma}_y^2(T/4) + \hat{\sigma}_y^2(T/2)\right) \neq \hat{\sigma}_{std}^2(T),$$

in contrast to (14). Since the definition of the Allan variance contains one $\tau$-overlap, we can admit all possible overlaps to obtain an improved estimator in order to minimize its error bars. Known as the standard "max-overlap" Allan estimator [5] given as (2), it also departs from a tractable connection to the standard variance for the same reason as the original $\tau$-overlap estimator.

Functionals which depend on a time interval $\Delta t$ have such a strong connection to spectral functions that (14) is a "decomposition" of the sample standard variance and seems an appropriate jargon and so is commonly used. In this regard, decomposition of the standard variance is suited to frequency-domain analysis, and Totvar maintains a straightforward relationship with the sample standard variance. It abides by the conservation principle if we consider an *infinite* extension by reflection. This means that the virtual sequence generated by (8) and shown in Figure 1 recurs indefinitely [10]. Percival [19] was the first to point out that for the case in which Totvar is computed in power-of-2 increments above $T/2$ as estimated from data-run $T$ as in (7), a remaining portion, the sum of Totvar terms of all power-of-2 intervals $\tau > T/2$ for $\tau \to \infty$, adds to the usual multiresolution pyramid to precisely equal the standard sample variance. These leftover higher-order components are never actually reported but are an artifact of infinitely extending the orginal sequence. They can be regarded as the sum of 0-frequency aliases, a remaining "D.C." term to make up $\hat{\sigma}_{std}^2(T)$. Greenhall [10] coined the term Remvar($\frac{T}{2}$) to designate this portion. Totvar beyond $T$ soon drops to nearly zero, so the remaining portion above $T$ is generally very small. Nevertheless, Remvar accounts for this portion and was used in the proof of the decomposition of the standard variance. Summing all the familiar "power-of-2" $\tau$-values in a Totvar plot leads to exactly twice the standard sample variance much in the same way that integrating an estimate of a spectrum also yields the sample variance.

Knowledge that we can account for all variations in a data-run by its standard variance as "decomposed" in calculations of the sample Total variance is especially useful. For example at a long-term $\tau$-value of $T/2$, an equal remaining portion (Totvar($\frac{T}{2}$)=Remvar($\frac{T}{2}$)) would indicate that we have summarized completely the variations at $T/2$.

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## Appendix I

Consider a series $\{X_{ij}\}$ with $k$ intervals each having $n$ values and means $m_j, j = 1, ..., k$. Assume

$$\textstyle\sum_1^{nk} X = 0, \sum_{j=1}^{k} m_j = 0 \tag{15}$$

and put

$$X_{ij} = x_{ij} + m_j, \tag{16}$$

so that

$$\textstyle\sum_{i=1}^{m} x_{ij} = 0. \tag{17}$$

The standard variance of the data run is denoted $V = \frac{SS}{mk}$ where

$$SS = \textstyle\sum_{j=1}^{k}\sum_{i=1}^{n}(x_{ij} + m_j)^2. \tag{18}$$

$$SS = \textstyle\sum_{i=1}^{n} x_{i1}^2 + ... + \sum_{i=1}^{n} x_{ik}^2 + nm_1^2 + nm_k^2 \tag{19}$$

plus terms of the form

$$\textstyle\sum_{i=1}^{n} 2x_{ij}m_j = 2m_j \sum_{i=1}^{n} x_{ij} = 0.$$

$$V = \frac{1}{nk}SS = \frac{1}{k}\left[\textstyle\sum_{i=1}^{n}\frac{x_{i1}^2}{n} + ... + \sum_{i=1}^{n}\frac{x_{ik}^2}{n}\right]$$

$$+\frac{m_1^2}{k} + ... + \frac{m_k^2}{k} = \frac{1}{k}\textstyle\sum_{j=1}^{k}\hat{\varepsilon}_{ss,j}^2(\Delta t) + v_m = \overline{\hat{\varepsilon}_{ss,j}^2(\Delta t)} + v_m \tag{20}$$

where $\overline{\hat{\varepsilon}_{ss,j}^2(\Delta t)}$ is the mean of the interval variance $\hat{\varepsilon}_{ss,j}^2(\Delta t)$, and $v_m$ is the variance of the interval means $m_j$.