DRAFT REVISION OF IEEE STD 1139-1988 STANDARD DEFINITIONS OF PHYSICAL QUANTITIES FOR FUNDAMENTAL FREQUENCY AND TIME METROLOGY - RANDOM INSTABILITIES*

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Abstract - This is a draft revision of IEEE Std 1139-1988 Standard Definitions of Physical Quantities for Fundamental Frequency and Time Metrology. This draft standard covers the fundamental metrology for describing random instabilities of importance to frequency and time metrology. Quantities covered include frequency, amplitude, and phase instabilities; spectral densities of frequency, amplitude, and phase fluctuations; variances of frequency and phase fluctuations; time prediction; and confidence limits when estimating the variance from a finite data set. The standard unit of measure for characterizing phase and frequency instabilities in the frequency domain is $\mathcal{L}(f)$, defined as one half of the double-sideband spectral density of phase fluctuations. In the time domain, the standard unit of measure of frequency and phase instabilities is the fully overlapped Allan deviation $\sigma_y(\tau)$ or the fully overlapped modified Allan deviation Mod $\sigma_y(\tau)$.

1. Introduction

This is a draft revision of IEEE Std 1139-1988 Standard Definitions of Physical Quantities for Fundamental Frequency and Time Metrology which had been prepared by a previous SCC 27 consisting of Helmut Hellwig, Chairman; David Allan; Peter Kartaschoff; Jacques Vanier; John Vig; Gernot M.R. Winkler; and Nicholas Yannoni. Some sections of the 1988 standard have remained unchanged. This standard covers the fundamental metrology for describing random instabilities of importance to frequency and time metrology. Quantities covered include frequency, amplitude, and phase instabilities; spectral densities of frequency, amplitude, and phase fluctuations; variances of frequency and phase fluctuations; time prediction; and confidence limits when estimating the variance from a finite data set. In addition, recommendations are made for the reporting of measurements of frequency and phase instabilities, especially as regards the recording of experimental parameters, experimental conditions and calculation techniques. This standard also covers translation between the frequency domain and time domain terminology and provides an extensive bibliography of the relevant literature. Systematic instabilities, such as environmental effects and aging, are discussed in IEEE Std 1193-1994.

2. Measures of Frequency, Amplitude and Phase Instabilities

The instantaneous output voltage of a precision oscillator can be expressed as

$$V(t) = (V_o + \varepsilon(t)) \sin(2\pi\nu_o t + \phi(t)), \qquad \text{(Eq. 1)}$$

where V_o is the nominal peak voltage amplitude, $\varepsilon(t)$ is the deviation from the nominal amplitude, ν_o is the nominal frequency, and $\phi(t)$ is the phase deviation from the nominal phase $2\pi\nu_o t$. Figure 1 illustrates a signal with frequency, amplitude, and phase instabilities. As shown, frequency instability is the result of fluctuations in the period of oscillation. Fluctuations in the phase result in instability of the zero crossing. Fluctuations in the peak value of the signal (V_{peak}) result in amplitude instability.

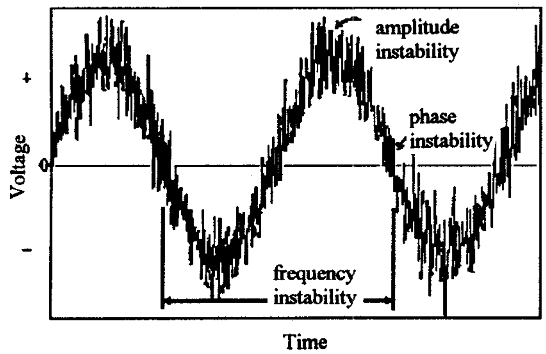

Figure 1. Instantaneous output voltage of an oscillator.[1]

Frequency instability of a precision oscillator is defined in terms of the instantaneous, normalized frequency deviation, y(t), as follows

$$y(t) = \frac{\nu(t) - \nu_o}{\nu_o} = \frac{\dot{\phi}(t)}{2\pi\nu_o}, \qquad \text{(Eq. 2)}$$

where $\nu(t)$ is the instantaneous frequency (time derivative of the phase divided by 2π), and

$$\dot{\phi}(t) = \frac{d\phi(t)}{dt}. \qquad \text{(Eq. 3)}$$

Amplitude instability is defined in terms of the instantaneous, normalized amplitude deviation

$$a(t) = \varepsilon(t)/V_o. \qquad \text{(Eq. 4)}$$

[1] In the signal shown the frequency components of the noise are higher than the carrier frequency. This is for illustration purposes only. In general, this standard applies to the frequency components of amplitude, phase and frequency instabilities which are lower in frequency than the carrier frequency.

Phase instability, defined in terms of the instantaneous phase deviation $\phi(t)$, can also be expressed in units of time, as

$$x(t) = \phi(t)/2\pi\nu_o. \qquad \text{(Eq. 5)}$$

With this definition, the instantaneous, normalized frequency deviation is

$$y(t) = dx(t)/dt. \qquad \text{(Eq. 6)}$$

Other random phenomena observed in certain oscillators are frequency jumps, that is, discontinuities in the frequency of oscillation. These phenomena are not repetitive or well understood and thus cannot be characterized by standard statistical methods.

3. Characterization of Frequency, Amplitude and Phase Instabilities

3.1 Frequency Domain. In the frequency domain, frequency, amplitude and phase instabilities can be defined or measured by one-sided spectral densities.

The unit of measure of frequency instability is the spectral density of fractional frequency fluctuations, $S_y(f)$, given by

$$S_y(f) = y^2(f)\frac{1}{BW}, \qquad \text{(Eq. 7)}$$

where y(f) is the root mean squared fractional frequency deviation as a function of Fourier frequency, BW is the measurement system bandwidth in Hz, and the units of $S_y(f)$ are 1/Hz.

The unit of measure of amplitude instability is the spectral density of fractional amplitude fluctuations, $S_a(f)$, given by

$$S_a(f) = \left(\frac{\varepsilon(f)}{V_o}\right)^2 \frac{1}{BW}. \qquad \text{(Eq. 8)}$$

The units of $S_a(f)$ are 1/Hz.

Phase instability can be characterized by the spectral density of phase fluctuations, $S_\phi(f)$, given by

$$S_\phi(f) = \phi^2(f)\frac{1}{BW}. \qquad \text{(Eq. 9)}$$

The units of $S_\phi(f)$ are rad^2/Hz.

These spectral densities, $S_y(f)$, $S_a(f)$, and $S_\phi(f)$, are one-sided since the Fourier frequency f ranges from 0 to ∞; nevertheless, they include fluctuations from both the upper and the lower sidebands of the carrier.

$S_\phi(f)$ is the quantity that is generally measured in frequency metrology; however, $\mathcal{L}(f)$ (pronounced *script ell* of f) has become the prevailing measure of phase noise among manufacturers and users of frequency standards. According to the *old* definition [Kartaschoff, 1978], $\mathcal{L}(f)$ is the ratio of the power in one sideband due to phase modulation by noise (for a 1 Hz bandwidth) to the total signal power (carrier plus sidebands); that is,

$$\mathcal{L}(f) = \frac{\text{power density in one phase noise modulation sideband, per Hz}}{\text{total signal power}}. \qquad \text{(Eq. 10)}$$

Usually $\mathcal{L}(f)$ is expressed in decibels (dB) as 10 log ($\mathcal{L}(f)$), and its units are dB below the carrier in a 1 Hz bandwidth, generally abbreviated as dBc/Hz.

The *old* definition of $\mathcal{L}(f)$ is related to $S_\phi(f)$ by

$$\mathcal{L}(f) \cong \frac{S_\phi(f)}{2}. \qquad \text{(Eq. 11)}$$

This definition breaks down when the mean squared phase deviation, $\langle\phi^2(t)\rangle$ = the integral of $S_\phi(f)$ from f to ∞, exceeds about 0.1 rad^2 or whenever there is a correlation between the power in the upper and lower sidebands. To circumvent this difficulty, $\mathcal{L}(f)$ is redefined as

$$\mathcal{L}(f) \equiv \frac{S_\phi(f)}{2}. \qquad \text{(Eq. 12)}$$

This redefinition is intended to avoid difficulties in the use of $\mathcal{L}(f)$ in situations where the small angle approximation is not valid and the correlation between the upper and lower sidebands is not measured. $\mathcal{L}(f)$, as defined by Eq. (12), is hereby designated as the standard measure of phase instability in the frequency domain. **The reasons are that (1) it can always be measured unambiguously, and (2) it conforms to the prevailing usage.**

Phase instability has sometimes been expressed in units of time by $S_x(f)$, the one-sided spectral density of the phase fluctuations expressed in units of time (x(t)),

$$S_x(f) = x^2(f)\frac{1}{BW}. \qquad \text{(Eq. 13)}$$

From Eq. (5), $S_\phi(f)$ and $S_x(f)$ are related by

$$S_x(f) = \frac{1}{(2\pi\nu_0)^2} S_\phi(f). \qquad \text{(Eq. 14)}$$

Since phase and frequency are directly related, that is, angular frequency is the time derivative of the phase, spectral densities of frequency and phase instabilities are also related:

$$S_y(f) = \frac{f^2}{\nu_0^{\,2}} S_\phi(f). \qquad \text{(Eq. 15)}$$

A device or signal should be characterized by a plot of $\mathcal{L}(f)$. In some applications, providing $\mathcal{L}(f)$ vs. discrete values of Fourier frequency is sufficient. (See Appendix A for further discussion.)

Other quantities related to phase instability are phase jitter and wander. Phase jitter is the integral of $S_\phi(f)$ over the Fourier frequencies of the application (usually above 10 Hz). When reporting phase jitter the range of Fourier frequencies and the slope of the discriminator used should be specified. Wander refers to the integral of $S_\phi(f)$ at Fourier frequencies below 10 Hz. It is possible to convert $S_\phi(f)$ to phase jitter by using the relation

$$\phi_{jitter}^2 = \int_{f_1}^{f_2} S_\phi(f)\,df, \qquad \text{(Eq. 16)}$$

where ϕ_{jitter}^2 refers to the phase jitter. It is not possible to obtain $S_\phi(f)$ from phase jitter, unless the shape of $S_\phi(f)$ is known.

<u>3.2 Time-Domain.</u> In the time domain, sequential average frequency instabilities are defined by a two-sample deviation $\sigma_y(\tau)$, also called the Allan deviation, which is the square root of a two-sample variance $\sigma_y^2(\tau)$, also called the Allan variance. This variance $\sigma_y^2(\tau)$ has no account of dead-time between adjacent frequency samples. (Dead time refers to the time between time-ordered data sets when no

measurement of frequency is taken.) For the sampling interval τ

$$\sigma_y(\tau) = \left[\frac{1}{2}\left\langle(\bar{y}(t+\tau)-\bar{y}(\tau))^2\right\rangle\right]^{1/2} = \left[\frac{1}{2}\left\langle(\bar{y}_{k+1}-\bar{y}_k)^2\right\rangle\right]^{1/2}, \quad \text{(Eq. 17)}$$

where

$$\bar{y}_k = \frac{1}{\tau}\int_{t_k}^{t_k+\tau} y(t)dt = \frac{x(t_k+\tau)-x(t_k)}{\tau} = \frac{x_{k+1}-x_k}{\tau}. \quad \text{(Eq. 18)}$$

The symbol < > denotes an infinite time average, and τ is the sampling interval. In practice, the requirement of infinite time average is never fulfilled, and the Allan deviation is estimated by

$$\sigma_y(\tau) \cong \left[\frac{1}{2(M-1)}\sum_{k=1}^{M-1}(\bar{y}_{k+1}-\bar{y}_k)^2\right]^{1/2}, \quad \text{(Eq. 19)}$$

where M is the number of frequency measurements.

The Allan deviation can also be expressed in terms of time difference (or time residual) measurements by combining Eqs. 18 and 19:

$$\sigma_y(\tau) \cong \left[\frac{1}{2(N-2)\tau^2}\sum_{k=1}^{N-2}(x_{k+2}-2x_{k+1}+x_k)^2\right]^{1/2}, \quad \text{(Eq. 20)}$$

where x_k, x_{k+1}, and x_{k+2} are time residual measurements made at t_k, $t_{k+1} = t_k+\tau$, and $t_{k+2} = t_k+2\tau$, k=1, 2, 3, ..., N is the number of time measurements, and $1/\tau$ is the nominal fixed sampling rate which gives zero dead time between frequency measurements. *Residual* implies the consistent, systematic effects such as frequency drift have been removed (see section 3.3).

If there is dead time between the frequency departure measurements and it is ignored in the computation of $\sigma_y(\tau)$, resulting instability values will be biased (except for white frequency noise). Some of the biases have been studied and some correction tables published [Barnes, 1969; Lesage, 1983; Barnes and Allan, 1988]. Therefore, the term $\sigma_y(\tau)$ shall not be used to describe such biased measurements without stating the bias together with $\sigma_y(\tau)$. The unbiased $\sigma_y(\tau)$ can be calculated from the biased, using information in the references. Considering that $\{x_k\}$ can be routinely measured, it is preferred that $\{x_k\}$ is used to compute $\sigma_y(\tau)$ since the problem of dead-time is solved.

If the initial sampling rate is specified as $1/\tau_o$, then, in general, we may obtain an estimate of $\sigma_y(\tau)$ with better confidence using what is called *overlapping estimates*. This estimate is obtained by computing

$$\sigma_y(\tau) = \left[\frac{1}{2(N-2m)\tau^2}\sum_{k=1}^{N-2m}(x_{k+2m}-2x_{k+m}+x_k)^2\right]^{1/2}, \quad \text{(Eq. 21)}$$

where N is the number of original time residual measurements spaced by τ_o (N = M + 1, where M is the number of original frequency measurements of sample time τ_o) and $\tau = m\tau_o$. Examples of overlapped $\sigma_y(\tau)$ estimates are given in Appendix B.

Equation 21 shows that $\sigma_y(\tau)$ acts like a second-difference operator on the time deviation residuals usually providing a stationary measure of the stochastic behavior even for nonstationary processes. An efficient spacing of τ values in a plot of $\log[\sigma_y(\tau)]$ vs. $\log[\tau]$ sets $m = 2^p$, where p = 0, 1, 2, 3, ...

The frequency stability of an oscillator should be characterized by a plot of $\sigma_y(\tau)$ vs. sampling time (τ). In some applications, providing discrete values of $\sigma_y(\tau)$ vs. sampling time τ is sufficient. The measurement system bandwidth should always be specified. (See Appendix A for further discussion.)

When differentiating between white and flicker PM noise is desirable, a *modified deviation*, denoted as Mod $\sigma_y(\tau)$, may be used to characterize frequency instabilities [Allan and Barnes, 1981; Stein, 1985]. Unlike $\sigma_y(\tau)$, Mod $\sigma_y(\tau)$ has the property of yielding different dependence on τ for white phase noise and flicker phase noise; the dependencies are $\tau^{-3/2}$ and τ^{-1}, respectively. (The dependence for $\sigma_y(\tau)$ is τ^{-1} for both white and flicker phase noise.) Another advantage is that Mod $\sigma_y(\tau)$ averages wideband phase noise faster than τ^{-1}. Mod $\sigma_y(\tau)$ is defined as

$$\text{Mod}\,\sigma_y(\tau) = \{\frac{1}{2\tau^2 m^2 (N-3m+1)} \sum_{j=1}^{N-3m+1} \left[\sum_{i=j}^{m+j-1} (x_{i+2m} - 2x_{i+m} + x_i) \right]^2 \}^{1/2}. \quad \text{(Eq. 22)}$$

For examples of $\sigma_y(\tau)$ and Mod $\sigma_y(\tau)$ see Appendix B.

A unit of measure that is often used in time transfer systems, such as GPS, is $\sigma_x(\tau)$. $\sigma_x(\tau)$ is defined as

$$\sigma_x(\tau) = \frac{\tau}{\sqrt{3}} \text{Mod}\,\sigma_y(\tau). \quad \text{(Eq. 23)}$$

This quantity is useful when white and flicker of phase modulation noise dominate a synchronization system.

Another approach to distinguish different noise types is to use multivariance analysis [Vernotte, et al., 1993]. By using several variances to analyze the same data it is possible to estimate the coefficients for the 5 noise types. For a description of noise types see Appendix A.

At long averaging times, when the number of samples is small ($N \leq 5$), $\sigma_y(\tau)$ has a bias related to its insensitivity to noise processes which appear as "odd" or antisymmetric functions over a finite measurement of x(t) (odd with respect to the middle point of the data). An extension of the Allan deviation ($\hat{\sigma}_{y,\text{TOTAL}}(\tau)$) which circularizes the original data provides a better estimate of frequency stability which is most noticeable at long averaging times and is recommended [Howe, 1995]. Details on $\hat{\sigma}_{y,\text{TOTAL}}(\tau)$ are given in Appendix B.

3.3 Systematic Instabilities. The long-term frequency change of a source is called frequency drift. Drift includes frequency changes caused by changes in the components of the oscillator, in addition to sensitivities to the oscillator's changing environment and changes caused by load and power supply changes [Vig and Meeker, 1991].

The frequency aging of an oscillator refers to the change in the frequency of oscillation caused by changes in the components of the oscillator, either in the resonant unit or in the accompanying electronics. Aging differs from drift in that it does not include frequency changes due to changes in the environment such as temperature changes. Aging is thus a measure of the long-term stability of the oscillator, independent of its environment. The frequency aging of a source (positive or negative) is typically maximum immediately after turn-on.

Aging can be specified by the normalized rate of change in frequency at a specified time after turn-on (for example, 1×10^{-10} per day after 30 days), or by the total normalized change in frequency in a period of time (for example, 1×10^{-8} per month) [Vig and Meeker, 1991].

3.4 Clock-Time Prediction. The variation of the time difference between a real clock and an ideal uniform time scale, also known as time interval error TIE, observed over a time interval starting at time t_0 and ending at $t_0 + t$ is defined as

$$\text{TIE}(t) = x(t_0 + t) - x(t_0) = \int_{t_0}^{t_0+t} y(t')dt'. \quad \text{(Eq. 24)}$$

For fairly simple models, regression analysis can provide efficient estimates of the TIE [Draper and Smith, 1966; CCIR, 1986]. In general, there are many estimators possible for any statistical quantity. Ideally, we would like an efficient and unbiased estimator. Using the time domain measure $\sigma_y(\tau)$ defined in section 3.2, the following estimate of the standard deviation (RMS) of TIE and its associated systematic departure due to a linear frequency drift (plus its uncertainty) can be used to predict a probable time interval error of a clock synchronized at $t = t_0 = 0$ and left free running thereafter

$$\text{RMS TIE}_{\text{est}}(t) = t(\left[\frac{x(t_0)}{t}\right]^2 + \sigma_{y_0}^2 + \sigma_y^2(\tau = t) + \frac{a^2}{4}t^2)^{1/2}, \quad \text{(Eq. 25)}$$

where

$x(t_0)$ = initial synchronization uncertainty,

σ_{y_0} = two-sample deviation of the initial frequency adjustment,

$\sigma_y(\tau)$ = the two-sample deviation describing the random frequency instability of the clock at $\tau = t$, a = normalized linear frequency drift per unit of time plus the uncertainty in the drift estimate.

The third term in the brackets provides an optimum and unbiased estimate (under the condition of an optimum (RMS) prediction method) in the cases of white noise FM and/or random walk FM. The third term is too optimistic, by about a factor of 1.4, for flicker noise FM, and too pessimistic, by about a factor of 3, for white noise PM.

This estimate is a useful and fairly simple approximation. A more complete error analysis becomes difficult; if carried out, such an analysis needs to include the methods of time prediction, the uncertainties of the clock parameters using the confidence limits of measurements defined below, the detailed clock noise models, systematic effects, etc.

A quantity often used to characterize the stability of clocks in telecommunication systems is the maximum time interval error (MTIE). MTIE is defined as the maximum time difference between a clock and an ideal reference [Bregni, 1996].

4. Confidence Limits of Measurements

A simple method to compute the confidence interval for $\sigma_y(\tau)$ [Lesage and Audoin, 1973], which assumes a symmetric (Gaussian) distribution, uses the relation

$$I_\alpha \cong \sigma_y(\tau)\kappa_\alpha M^{-1/2}, \qquad \text{(Eq. 26)}$$

where I_α is the confidence interval, κ is a constant, α is an integer that depends on the type of noise (see Appendix A), and M is the total number of data points used in the estimate. For a 1σ or 68 % confidence interval the values for κ_α are

$\kappa_2 = 0.99$,
$\kappa_1 = 0.99$,
$\kappa_0 = 0.87$,
$\kappa_{-1} = 0.77$,
$\kappa_{-2} = 0.75$.

As an example of the Gaussian model with M = 100, $\alpha = -1$ (flicker frequency noise) and $\sigma_y(\tau = 1\text{ s}) = 1 \times 10^{-12}$, we may write

$$I_\alpha \cong \sigma_y(\tau)(0.77)(100)^{-1/2} = \sigma_y(\tau)\ (0.077),$$

which gives

$$\sigma_y(\tau = 1\text{ s}) = (1 \pm 0.08)\ 10^{-12}.$$

This analysis for $\sigma_y(\tau)$ is valid only for $M \geq 10$. If M is small, then the plus and minus confidence intervals become sufficiently asymmetric and the κ_α coefficients are not valid; however, these confidence intervals can be calculated [Lesage and Audoin, 1973].

Another way of computing confidence intervals for $\sigma_y(\tau)$ is to use the chi-squared distribution. The estimated Allan variance has a chi-squared distribution function given by Eq. 27. The number of degrees of freedom for a specific noise process and number of samples can be computed and then used in Eq. 27 to compute the confidence interval [Howe, 1981]:

$$\chi^2 = (\text{df})\frac{\hat{\sigma}_y^2}{\sigma_y^2}, \qquad \text{(Eq. 27)}$$

where df is the number of degrees of freedom, $\hat{\sigma}_y^2$ is the estimated (measured) Allan variance, and σ_y^2 is the true Allan variance. Table C1 (Appendix C) shows empirical equations to compute the number of degrees of freedom for different types of noise processes (N number of samples, and $m = \tau/\tau_o$). Examples of this method are shown in Appendix C.

Other methods have been developed for calculating confidence intervals for Mod $\sigma_y(\tau)$. For a detailed description see [Walter, 1994; Greenhall, 1995]. Comparison of confidence intervals for $\sigma_y(\tau)$ (no-overlap, full-overlap) and Mod $\sigma_y(\tau)$ are given in Appendix C (Table C2)].

5. Standards for Characterizing or Reporting Measurements of Frequency and Phase Instabilities

The standard unit of measure for characterizing phase and frequency instabilities in the frequency domain is $\mathcal{L}(f)$, defined as one half of the double-sideband spectral density of phase fluctuations (see Eq. 12). When expressed in decibels its units are dBc/Hz (dB below the carrier in a 1 Hz bandwidth). A device should be characterized by a plot of $\mathcal{L}(f)$ versus f. In the time domain, the standard unit of measure of frequency and phase instabilities is the fully overlapped Allan deviation $\sigma_y(\tau)$ or the fully overlapped modified Allan deviation Mod $\sigma_y(\tau)$. A device should be characterized by a plot of $\sigma_y(\tau)$ or Mod $\sigma_y(\tau)$ versus τ. The measurement system bandwidth (f_h) and the total measurement time should be indicated.

In addition the following provisions are recommended when reporting measurements on frequency and phase instabilities.

5.1 Nonrandom phenomena should be recognized; for example:

(1) Any observed time dependence of the statistical measures should be stated.
(2) The method of modeling systematic behavior should be specified (for example, an estimate of the linear frequency drift was obtained from the coefficients of a linear least-squares regression to M frequency measurements, each with a specified averaging or sample time t and measurement bandwidth f_h).
(3) The environmental sensitivities should be stated (for example, the dependence of frequency and/or phase on temperature, magnetic field, barometric pressure, vibration, etc.).

5.2 Relevant measurement or specification parameters should be given:

(4) The method of measurements.
(5) The characteristics of the reference signal (equal noise or much lower noise assumed).
(6) The nominal signal frequency ν_o.
(7) The measurement system bandwidth f_h and the corresponding low pass filter response.
(8) The total measurement time and the number of measurements M.
(9) The calculation techniques (for example, details of the window function when estimating power spectral densities from time domain data, or the assumptions about effects of dead time when estimating the two-sample deviation $\sigma_y(t)$).
(10) The confidence of the estimate (or error bar) and its statistical probability (for example, *1σ for 68%, 2σ for 95%*).
(11) The environment during measurement.
(12) If a passive element, such as a crystal filter, is being measured in contrast to a frequency and/or time generator.

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7. References

[1] Allan, D.W., and Barnes, J.A., "A Modified "Allan Variance" with Increased Oscillator Characterization Ability," Proc. 35th Annual Freq. Control Symp., pp. 470-474, 1981.

[2] Barnes, J.A., "Tables of Bias Functions, B1 and B2, for Variances Based on Finite Samples of Processes with Power Law Spectral Densities," Nat. Bur. Stands. (U.S.) Tech. Note 375, Washington DC, Jan. 1969.

[3] Barnes J.A., and Allan, D.W., "Variances Based on Data with Dead-Time Between the Measurements: Theory and Tables," Nat. Bur. Stands. (U.S.) Tech. Note 1318, Washington DC, 1988.

[4] Barnes, J.A., Chi, A.R., Cutler, L.S., Healey, D.J., Leeson, D.B., Mc Gunigal, T.E., Mullen, J.A., Smith, W.L., Sydnor, R., Vessot, R.F., and Winkler, G.M.R., "Characterization of Frequency Stability," IEEE Trans. Instrum. Meas., Vol. IM-20, 105-120, May 1971.

[5] Bregni, S.R., "Measurement of Maximum Time Interval Error for Telecommunications Clock Stability Characterization," IEEE Trans. Instrum. Meas., Vol. IM-45, 900-905, Oct., 1996.

[6] Draper N.R., and Smith., H., Applied Regression Analysis, New York: John Wiley and Sons, 1986.

[7] Dubrovnik, CCIR Publication, Report 898, National Technical Information Service, Department of Commerce, 5285 Port Royal Road, Springfield, VA 22161, 1986.

[8] Greenhall, C.A., "Estimating the Modified Allan Variance," Proc. 1995 IEEE Int'l Freq. Control Symp., 346-353, June 1995.

[9] Howe, D.A., Allan, D.W., and Barnes, J.A., "Properties of Signal Sources and Measurement Methods," Proc. 35th Annual Symp. Freq. Control, 669-717, May 1981.

[10] Howe, D.A., "An Extension of the Allan Variance with Increased Confidence at Long Term," Proceedings 1995 IEEE Int'l Freq. Control Symp., 321-329, June, 1995.

[11] IEEE Std 1193-1994, IEEE Guide for Measurements of Environmental Sensitivities of Standard Frequency Generators, IEEE Inc., New York, 1994.

[12] Kartaschoff, P., Frequency and Time, Academic Press, New York, 1978.

[13] Lesage, P., and Audoin, C., "Characterization of Frequency Stability: Uncertainty Due to the Finite Number of Measurements," IEEE Trans. Instrum. Meas., Vol. IM-22, 157-161, June 1973.

[14] Lesage, P., "Characterization of Frequency Stability: Bias Due to the Juxtaposition of Time Interval Measurements," IEEE Trans. Instrum. Meas., IM- 32, 204-207, 1983.

[15] Stein, S.R., "Frequency and Time: Their Measurement and Characterization," Precision Frequency Control, Vol. 2, Academic Press, New York, 191-232, 1985.

[16] Sullivan, D.B., Allan, D.W., Howe, D.A., and Walls, F.L., Natl. Inst. Stand. Technol. Tech. Note 1337, Washington D.C., March 1990.

[17] Tuladhar, K.K., and Jenni, G., "Frequency Jumps on BVA and Other Precision Quartz Crystal Resonators, and Burst Noise on Overtone Mode High-Frequency Quartz Crystal Resonators," part 1 in the 10th European Frequency & Time Forum, (1996), and part II in the Proc. IEEE Int'l Freq. Control Symp. (1996).

[18] Vernotte, F., Lantz, E., Groslambert, J., and J.J. Gagnepain, J.J., "Oscillator Noise Analysis: Multi-Variance Measurement," IEEE Trans. Instrum. Meas., Vol. IM-42, No. 2, 342-350, April 1993.

[19] Vig, J.R., and Meeker, T.R., "The Aging of Bulk Acoustic Wave Resonators, Filters and Oscillators," Proc. 45th Annual Symp. Freq. Control, 77 - 101, 1991.

[20] Walter, T. "Characterizing Frequency Stability: A Continuous Power-Law Model with Discrete Sampling," IEEE Trans. Instrum. Meas., IM 43, 69-79, Feb. 1994.

Appendix A

Power-Laws and Conversion between Frequency and Time Domain

A1. Power-Law Spectral Densities. Power-law spectral densities serve as reasonable and accurate models of the random fluctuations in precision oscillators. In practice, these random fluctuations can often be represented by the sum of five such noise processes assumed to be independent, as

$$S_y(f) = \begin{cases} \sum_{\alpha=-2}^{+2} h_\alpha f^\alpha & \text{for } 0 < f < f_h, \\ 0 & \text{for } f \geq f_h, \end{cases} \qquad \text{(Eq. A1)}$$

where

h_α = constant,
α = integer,
f_h = high-frequency cut-off of an infinitely sharp low pass filter.

High frequency divergence is eliminated by the restrictions on f in this equation. The identification and characterization of the five noise processes are given in Table A1, and shown in Fig. A1.

A2. Conversion between Frequency and Time Domain. The operation of the counter, averaging the frequency for a time τ, may be thought of as a filtering operation. The transfer function H(f) of this equivalent filter is then the Fourier transform of the impulse response of the filter. The time domain frequency instability is then given by

$$\sigma^2(M,T,\tau) = \int_0^\infty S_y(f)\left|H(f)\right|^2 df, \qquad \text{(Eq. A2)}$$

where $S_y(f)$ is the spectral density of frequency fluctuations. 1/T is the measurement rate ($T-\tau$ is the dead time between measurements). In the case of the two-sample variance $|H(f)|^2$ is $2(\sin^4\pi\tau f)/(\pi\tau f)^2$. The two-sample variance can thus be computed from

$$\sigma_y^2(\tau) = 2\int_0^{f_h} S_y(f)\frac{\sin^4(\pi\tau f)}{(\pi\tau f)^2}df. \qquad \text{(Eq. A3)}$$

Specifically, for the power law model given, the time domain measure also follows a power law:

$$\sigma_y^2(\tau) = h_{-2}\frac{(2\pi)^2}{6}\tau + h_{-1}2\log_e 2 + h_0\frac{1}{2\tau}$$
$$+h_1\frac{1.038+3\log_e(2\pi f_h\tau)}{(2\pi)^2\tau^2} + h_2\frac{3f_h}{(2\pi)^2\tau^2}. \quad \text{(Eq.A4)}$$

Equation A4 assumes that f_h is the high frequency cut-off of an infinitely sharp low pass filter and that $2\pi f_h\tau >> 1$. This equation also implicitly assumes that the random driving mechanism for each term is independent of the others, and that the mechanism is valid over all Fourier frequencies, which may not always be true.

The values of h_α are characteristic models of oscillator frequency noise. For integer values (as often seems to be the case for reasonable models),

$\mu = -\alpha-1$, for $-3 \le \alpha \le 1$,
$\mu \approx -2$ for $\alpha \ge 1$,

where

$$\sigma_y^2(\tau) \sim \tau^\mu.$$

The modified two-sample variance can also be computed from $S_y(f)$ by using

$$\text{Mod}\sigma_y^2(\tau) = \frac{2}{n^4}\int_0^{f_h} S_y(f)\frac{\sin^6(\pi\tau f)}{(\pi\tau_0 f)^2\sin^2(\pi\tau_0 f)}df. \quad \text{(Eq. A5)}$$

Table A2 gives the coefficients of the translation from $S_y(f)$ (frequency domain) to $\sigma_y^2(\tau)$ (time domain). In general computation of $S_y(f)$ or related frequency domain measurements from $\sigma_y(\tau)$ or Mod $\sigma_y(\tau)$ are not permitted unless only one power law noise type is present. Nevertheless, when several noise types are present, special analysis can be made on the time domain data to obtain the coefficients (in the frequency domain) for each power law [Vernotte, et al., 1993].

The slope characteristics of the five independent noise processes are plotted in the frequency and time domains in Fig. A1 (log-log scale).

Table A1
The Functional Characteristics of the Independent Noise Processes Used in Modeling Frequency Instability of Oscillators

Description of Noise Process	Slope Characteristics of Log Log Plot				
	Frequency Domain		Time-Domain		
	$S_y(f)$ or $S_{\dot\phi}(f)$	$S_\phi(f)$ or $S_x(f)$	$\sigma_y^2(\tau)$	$\sigma_y(\tau)$	Mod $\sigma_y(\tau)$
	α	β	μ	$\mu/2$	μ'
Random walk frequency modulation	-2	-4	1	1/2	1/2
Flicker frequency modulation	-1	-3	0	0	0
White frequency modulation	0	-2	-1	-1/2	-1/2
Flicker phase modulation	1	-1	-2	-1	-1
White phase modulation	2	0	-2	-1	-3/2

$$S_y(f) = \frac{(2\pi f)^2}{(2\pi\nu_0)^2}S_\phi(f) = h_\alpha f^\alpha \qquad \sigma_y^2(\tau) \sim |\tau|^\mu$$
$$S_\phi(f) = \nu_0^2 h_\alpha f^{\alpha-2} = \nu_0^2 h_\alpha f^\beta \quad (\beta \equiv \alpha - 2) \qquad \sigma_y(\tau) \sim |\tau|^{\mu/2}$$
$$S_x(f) = \frac{1}{4\pi^2}h_\alpha f^{\alpha-2} = \frac{1}{4\pi^2}h_\alpha f^\beta \qquad \text{Mod}\,\sigma_y(\tau) \sim |\tau|^{\mu'}$$

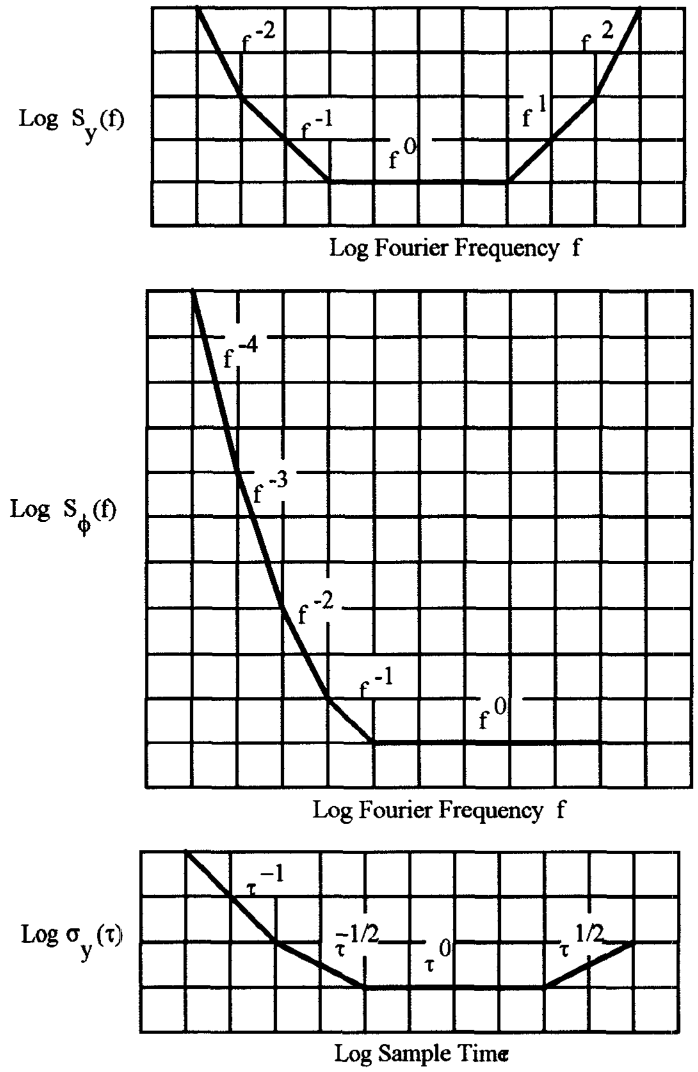

Fig. A1 Slope Characteristics of the Five Independent Noise Process

Table A2. Translation of Frequency Instability Measures from Spectral Densities in Frequency Domain to Variances in Time Domain for an Infinitely Sharp Low Pass Filter with $2\pi f_h \tau >> 1$.

Description of Noise Process	$S_y(f) =$	$S_\phi(f) =$	$\sigma_y^2(\tau) =$
Random walk frequency modulation	$h_{-2}f^{-2}$	$h_{-2}\nu^2 f^{-4}$	$Ah_{-2}\tau^{1}$
Flicker frequency modulation	$h_{-1}f^{-1}$	$h_{-1}\nu^2 f^{-3}$	$Bh_{-1}\tau^{0}$
White frequency modulation	$h_0 f^0$	$h_0\nu^2 f^{-2}$	$Ch_0\tau^{-1}$
Flicker phase modulation	$h_1 f^1$	$h_1\nu^2 f^{-1}$	$Dh_1\tau^{-2}$
White phase modulation	$h_2 f^2$	$h_2\nu^2 f^0$	$Eh_2\tau^{-2}$

$A = \dfrac{2\pi^2}{3}$ $\quad B = 2\ln 2$ $\quad C = 1/2$ $\quad D = \dfrac{1.038 + 3\ln(2\pi f_h \tau)}{4\pi^2}$ $\quad E = \dfrac{3f_h}{4\pi^2}$

Appendix B

Examples and Additional Variances that may be used to Describe Frequency Instabilities in the Time Domain

This appendix contains basic examples on how to compute the variances used to describe frequency instabilities in the time domain. For more information on this topic and on how to assess the validity of the computations when using larger number of samples see [Riley, 1995; Riley, 1996].

B1. $\sigma_y(\tau)$ - Examples. Fig. B1 shows a plot of the time deviation between a pair of oscillators as a function of time. The recorded time samples for τ = 1 s are shown in the first column of Table B1. To compute $\sigma_y(\tau = 1\ s)$ we need to compute the average fractional frequency deviation for x_ks separated by 1 s, then calculate the difference between adjacent $\bar{y}_k$s and use Eq. 19 to obtain $\sigma_y(\tau)$. See Table B1.

Table B1. Steps to compute $\sigma_y(1\ s)$.

x_k (10^{-6})	$\bar{y}_k = (x_{k+1} - x_k)/\tau$ (10^{-6})	$\bar{y}_{k+1} - \bar{y}_k$ (10^{-6})
0	--	--
43.6	$\bar{y}_1 = (x_2-x_1)/\tau = 43.6$	--
89.7	$\bar{y}_2 = (x_3-x_2)/\tau = 46.1$	2.5
121.6	$\bar{y}_3 = (x_4-x_3)/\tau = 31.9$	-14.2
163.7	$\bar{y}_4 = (x_5-x_4)/\tau = 42.1$	10.2
208.4	$\bar{y}_5 = (x_6-x_5)/\tau = 44.7$	2.6
248	$\bar{y}_6 = (x_7-x_6)/\tau = 39.6$	-5.1
289	$\bar{y}_7 = (x_8-x_7)/\tau = 41.0$	1.4
319.8	$\bar{y}_8 = (x_9-x_8)/\tau = 30.8$	-10.2

In this example N = 9 (number of time samples) and M = 8; therefore

$$\sigma_y(\tau = 1s) = \left[\frac{1}{2(7)}\sum_{k=1}^{7}(\bar{y}_{k+1} - \bar{y}_k)^2\right]^{1/2}, \quad \text{(Eq. B1)}$$

$$\sigma_y(1s) = [3.2\text{x}10^{-11}]^{1/2} = 5.67 \text{ x } 10^{-6}.$$

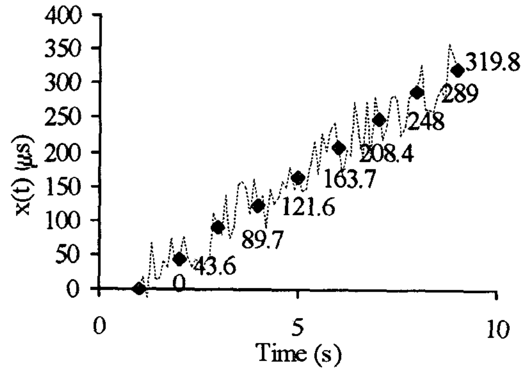

Figure B1. Plot of x(t) between a pair of oscillators.

For $\tau = 2$ s ($= 2\tau_o$) the procedure is similar: compute the fractional frequency deviation for x_ks separated by 2 s, then calculate the difference between adjacent $\bar{y}_k$s and use Eq. 19 to obtain $\sigma_y(\tau)$. See Table B2.

Table B2. Steps for computing $\sigma_y(2\ s)$.

x_k (10^{-6})	$\bar{y}_k = (x_{k+2} - x_k)/\tau$ (10^{-6})	$\bar{y}_{k+2} - \bar{y}_k$ (10^{-6})
0	--	--
43.6	--	--
89.7	$\bar{y}_1=(x_3-x_1)/2=44.85$	--
121.6	--	--
163.7	$\bar{y}_3=(x_5-x_3)/2=37$	-7.85
208.4	--	--
248	$\bar{y}_5=(x_7-x_5)/2=42.15$	5.15
289	--	--
319.8	$\bar{y}_7=(x_9-x_7)/2=35.9$	-6.25

For this example M = 4 since there are a total of four $\bar{y}_k$s. Therefore

$$\sigma_y(\tau = 2s) = \left[\frac{1}{2(3)}\sum_{k=1}^{3}(\bar{y}_{k+m} - \bar{y}_k)^2\right]^{\frac{1}{2}}, \quad \text{(Eq. B2)}$$

$$\sigma_y(2s) = [2.12\text{x}10^{-11}]^{\frac{1}{2}} = 4.6 \text{ x } 10^{-6}.$$

As mentioned in Section 3.2, it is usually more efficient to use overlapped estimates when possible since it results in a better confidence interval. Figure B2 illustrates how to compute the $\bar{y}_k$s for an overlapped estimate of $\sigma_y(\tau = 2\ s)$. In this case m = 2 ($\tau = 2\tau_o$) and M = 8. The $\bar{y}_k$s and the second difference values are shown in Table B3.

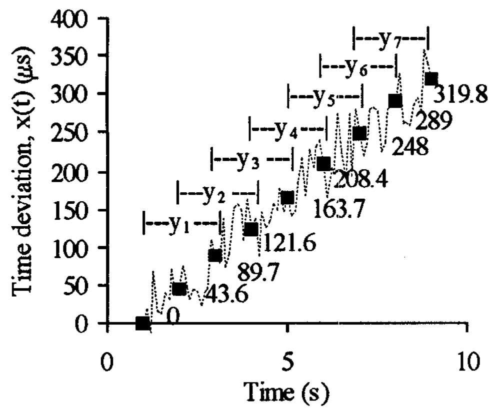

Figure B2. Computation of $\bar{y}_k$s for overlapped estimates.

There are a total of (N - 2m = 5) second difference values ($\bar{y}_{k+m}$-$\bar{y}_k$); therefore the Allan deviation equation becomes

$$\sigma_y(\tau) = \left[\frac{1}{2(N-2m)} \sum_{k=1}^{N-2m} (\bar{y}_{k+m} - \bar{y}_k)^2 \right]^{\frac{1}{2}}, \quad \text{(Eq. B3)}$$

where $\bar{y}_k = (x_{k+m} - x_k)/\tau$. Equation B3 becomes Eq. 21 when the $\bar{y}_k$s are expressed in terms of the initial time residual measurements. It is used in this example to help explain the origin of Eq. 21. Using the values in Table B3 (last column), Eq. B3 yields

$$\sigma_y(2s) = \{\frac{1}{2(5)}[(-7.85)^2 + 4.4^2 + 5.15^2 + (-3.1)^2 + (-6.25)^2]\}^{\frac{1}{2}},$$

$$\sigma_y(2s) = [1.56 \text{ x } 10^{-11}]^{\frac{1}{2}} = 3.95 \text{ x } 10^{-6}.$$

B2. Mod $\sigma_y(\tau)$ - Example. The Modified Allan deviation can also be used to characterize frequency stability in the time domain. This deviation uses the average of m adjacent x_ks when computing the stability for $\tau = m\tau_o$. The fractional frequency deviations are then obtained using the $\bar{x}_k$s. See Fig. B3 for computation of $\bar{x}_k$s and $\bar{y}_k$'s.

Table B4 shows the computed $\bar{x}_k$s, and $\bar{y}_k$'s for τ = 2s. The Modified Allan deviation can then be obtained by using Eq. 19 and the fact that m = 2 and the equivalent M is N - 3m + 1:

$$\text{Mod } \sigma_y(\tau) = \{\frac{1}{2(N-3m+1)} \sum_{k=1}^{N-3m+1} (\bar{y}'_{k+m} - \bar{y}'_k)^2\}^{\frac{1}{2}}, \quad \text{(Eq. B4)}$$

$$\text{Mod } \sigma_y(2s) = \{\frac{1}{2(4)}[(-1.73)^2 + 4.78^2 + 1.03^2 + (-4.68)^2]\}^{\frac{1}{2}} = [6.1\text{x}10^{-12}]^{\frac{1}{2}},$$

$$\text{Mod } \sigma_y(2\text{ s}) = 2.47\text{x}10^{-6}.$$

Note that Eq. B4 becomes Eq. 22 when expressing the $\bar{y}_k$'s in terms of the initial time residual measurements. It is used in this example to help explain the origin of Eq. 22.

Table B3. Steps for computing an overlapped estimate of σ_y(2 s).

x_k (10^{-6})	$\bar{y}_k = (x_{k+2} - x_k)/\tau$ (10^{-6})	$\bar{y}_{k+2} - \bar{y}_k$ (10^{-6})
0	--	--
43.6	--	--
89.7	$\bar{y}_1 = (x_3 - x_1)/\tau = 44.9$	--
121.6	$\bar{y}_2 = (x_4 - x_2)/\tau = 39$	--
163.7	$\bar{y}_3 = (x_5 - x_3)/\tau = 37$	-7.85
208.4	$\bar{y}_4 = (x_6 - x_4)/\tau = 43.4$	4.4
248	$\bar{y}_5 = (x_7 - x_5)/\tau = 42.2$	5.15
289	$\bar{y}_6 = (x_8 - x_6)/\tau = 40.3$	-3.1
319.8	$\bar{y}_7 = (x_9 - x_7)/\tau = 35.9$	-6.25

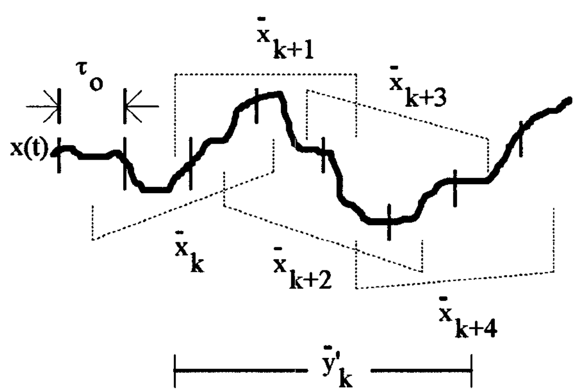

Figure B3. Method for calculating $\bar{x}_k$s, and $\bar{y}_k$'s for Mod $\sigma_y(\tau)$.

Table B4. Computed $\bar{x}_k$, and $\bar{y}_k'$ values for Mod $\sigma_y(2\text{ s})$.

x_k (10^{-6})	$\bar{x}_k = (x_{k+1}+x_k)/2$ (10^{-6})	$\bar{y}_k' = (\bar{x}_{k+2} - \bar{x}_k)/\tau$ (10^{-6})	$\bar{y}_{k+2}'-\bar{y}_k'$ (10^{-6})
0	--	--	--
43.6	21.8	--	--
89.7	66.65		--
121.6	105.65	$\bar{y}_1' = (\bar{x}_3 - \bar{x}_1)/2 = 41.93$	--
163.7	142.65	$\bar{y}_2' = (\bar{x}_4 - \bar{x}_2)/2 = 38$	--
208.4	186.05	$\bar{y}_3' = (\bar{x}_5 - \bar{x}_3)/2 = 40.2$	-1.73
248	228.2	$\bar{y}_4' = (\bar{x}_6 - \bar{x}_4)/2 = 42.78$	4.78
289	268.5	$\bar{y}_5' = (\bar{x}_7 - \bar{x}_5)/2 = 41.23$	1.03
319.8	304.4	$\bar{y}_6' = (\bar{x}_8 - \bar{x}_6)/2 = 38.1$	-4.68

B3. Extension of the Allan Deviation $\hat{\sigma}_{y,\text{TOTAL}}(\tau)$. In section 3.2 we noted that when the number of samples is small ($N \le 5$), the Allan deviation has a bias related to its insensitivity to odd (antisymmetric) noise processes in x(t) (odd about the midpoint of the x(t) data or even in terms of average $\bar{y}_k$). This is illustrated in Fig. B4. Figure B4.1 shows three phase samples of a noise process which is odd about x_2. The calculated fractional frequency deviations according to Eq. 19 are shown in Fig. B4.2. Since $\bar{y}_1$ and $\bar{y}_2$ are equal, contributions due to this noise process will not show up in $\sigma_y(\tau)$.

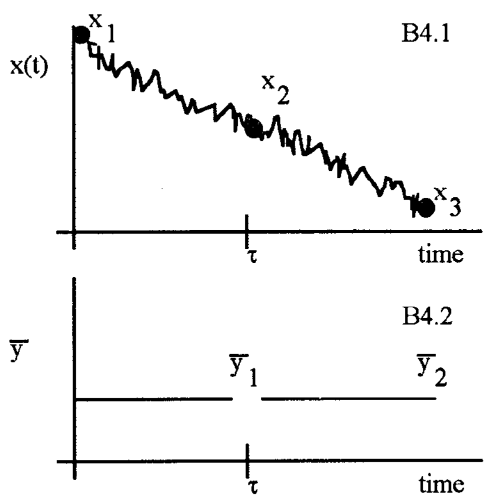

Figure B4. Odd noise process about x_2.

For this reason, when $N \le 5$, we recommend using an extension of the 2-sample (Allan) deviation which adds the odd (3-sample) component [Howe, 1995]. The estimate to this deviation, $\hat{\sigma}_{y,\text{TOTAL}}(\tau)$, wraps the x_k data to provide a better estimate of frequency stability. The advantages of $\hat{\sigma}_{y,\text{TOTAL}}(\tau)$ are outlined in several references [Howe, 1995; Howe, 1997] and its generalized form is given by

$$\hat{\sigma}_{y,\text{TOTAL}}(\tau) = \left(\frac{1}{\tau^2}\frac{1}{2(N-m-1)}\right)^{1/2} \left(\sum_{k=1}^{N-m-1} [x'_{k+2m} - 2x'_{k+m} + x'_k]^2\right)^{1/2}, \quad \text{(Eq. B5)}$$

where the x'_ks represent the time residual measurements after the linear trend (or slope in phase) has been removed (matching the endpoints of the data set $\{x_k\}$). $\hat{\sigma}_{y,\text{TOTAL}}(\tau)$ can also be represented in terms of fractional frequency fluctuation averages as

$$\hat{\sigma}_{y,\text{TOTAL}}(\tau) = \left\{\frac{1}{N-2m+1}\sum_{j}^{N-2m+1} \left[\frac{1}{2(N-2m)}\sum_{k=1}^{N-2m} (\bar{y}_{k+m,j} - \bar{y}_{k,j})^2\right]\right\}^{\frac{1}{2}}, \quad \text{(Eq. B6)}$$

where the series of $\bar{y}_{k,j}$ (fractional frequency fluctuations averages) goes as $\bar{y}_{j+1}, \bar{y}_{j+2}, \ldots, \bar{y}_{N-1}, \bar{y}_1, \ldots, \bar{y}_j$.

As an example we will compute $\hat{\sigma}_{y,TOTAL}(2s)$ for x(t) in Fig. B5. Notice that this data set is different from the one used in the previous examples. In this case x_1, x_3, and x_5 almost fall into a line; therefore the value for $\sigma_y(\tau)$ will be negatively biased (too optimistic).

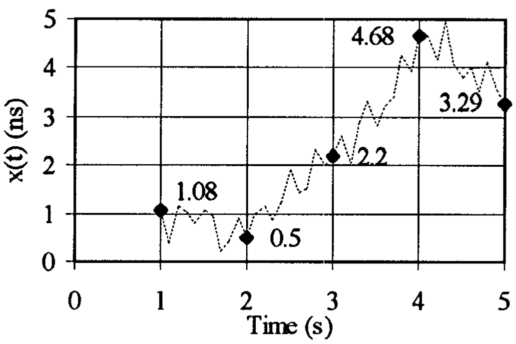

Figure B5. x(t) as a function of time.

Starting with the time difference measurements $\{x_k\}$, the first step is to compute the overall frequency difference (OFD = $(x_N - x_1)/(N-1)\tau_o$) and remove it from each x_k, the result being that x_1 and x_5 will have the same value. This is necessary in order to be able to wrap the data without a discontinuous phase step ($x_5 \rightarrow x_1$). Therefore,

$$\text{OFD} = \frac{x_5 - x_1}{4} = \frac{3.29 - 1.08}{4} = 0.5525\text{x}10^{-9}. \quad \text{(Eq. B7)}$$

Now remove the OFD from x_ks:

$x'_1 = 1.08$ ns,
$x'_2 = 0.5 - \text{OFD} = -0.0525$ ns,
$x'_3 = 2.2 - 2(\text{OFD}) = 1.095$ ns,
$x'_4 = 4.68 - 3(\text{OFD}) = 3.0225$ ns,
$x'_5 = 3.29 - 4(\text{OFD}) = 1.08$ ns.

$\hat{\sigma}_{y,TOTAL}(2s)$ can then be computed using Eq. B8:

$$\hat{\sigma}_{y,TOTAL}(2\,\text{sec}) = \{\frac{1}{2(2)}[(\bar{y}_{2,1} - \bar{y}_{1,1})^2 + (\bar{y}_{2,2} - \bar{y}_{1,2})^2]\}^{\frac{1}{2}}, \quad \text{(Eq. B8)}$$

where

$$\bar{y}_{1,1} = \frac{x'_3 - x'_1}{2} = \frac{1.095 - 1.08}{2} = 0.0075\text{x}10^{-9},$$

$$\bar{y}_{2,1} = \frac{x'_5 - x'_3}{2} = \frac{1.08 - 1.095}{2} = -0.0075\text{x}10^{-9},$$

$$\bar{y}_{1,2} = \frac{x'_4 - x'_2}{2} = \frac{3.0225 + 0.0525}{2} = 1.5375\text{x}10^{-9},$$

$$\bar{y}_{2,2} = \frac{x'_2 - x'_4}{2} = \frac{-0.0525 - 3.0225}{2} = -1.5375\text{x}10^{-9}.$$

Therefore,

$$\hat{\sigma}_{y,TOTAL}(2s) = \{\frac{1}{4}[(-0.015)^2 + (-3.075)^2]\text{x}10^{-18}\}^{\frac{1}{2}}$$

$$= [2.364 \text{ x } 10^{-18}]^{\frac{1}{2}},$$

$$\hat{\sigma}_{y,TOTAL}(2s) = 1.54 \text{ x } 10^{-9}.$$

This value can be compared to the value obtained for the Allan deviation:

$$\sigma_y(2s) = \{\frac{1}{2}[\bar{y}_3 - \bar{y}_1]^2\}^{\frac{1}{2}}$$

$$= \{\frac{1}{2}[\frac{x_5 - 2x_3 + x_1}{2}]^2\}^{\frac{1}{2}}$$

$$= \{\frac{1}{8}[3.29 - 2(2.2) + 1.08)]^2\}^{\frac{1}{2}},$$

$$\sigma_y(2\text{ s}) = 1.06 \text{ x } 10^{-11}.$$

$\sigma_y(2$ s) is seriously negatively biased by two orders of magnitude compared to $\hat{\sigma}_{y,TOTAL}(2s)$. (In the computation of the Allan deviation the OFD was not subtracted from the x_ks since the Allan deviation is insensitive to linear trends in the time domain).

A slight negative bias in $\hat{\sigma}_{y,TOTAL}(\tau)$ has been found for flicker FM noise and random walk FM noise. The bias for random walk FM noise can be removed by adjusting Eq. B5 [Howe, 1997].

<u>B4. Other Variances.</u> Several other variances have been introduced by workers in this field. In particular, before the introduction of the two-sample variance, it was standard practice to use the sample variance s^2, defined as

$$s^2 = \int_0^{f_h} S_y(f)\left(\frac{\sin \pi f\tau}{\pi f\tau}\right)^2 df. \qquad \text{(Eq. B9)}$$

In practice it may be obtained from a set of measurements of the frequency of the oscillator as

$$s^2 = \frac{1}{N}\sum_{i=1}^{N}(y_i - \bar{y})^2. \qquad \text{(Eq. B10)}$$

The sample variance diverges for some types of noise and, therefore, is not generally useful.

Other variances based on the structure function approach can also be defined [Lindsey and Chi, 1976]. For example, there are the Hadamard variance, the three-sample variance and the high-pass variance [Rutman 1978]. They are occasionally used in research and scientific works for specific purposes, such as differentiating between different types of noise and for dealing with systematics and sidebands in the spectrum.

Appendix C

Confidence Intervals for $\sigma_y(\tau)$ - Examples

As discussed in section 4, the Allan variance has a chi-squared distribution function. Therefore, one way to obtain the confidence interval for a certain number of samples is to use

$$\chi^2 = (df)\frac{\hat{\sigma}_y^2}{\sigma_y^2}, \qquad \text{(Eq. C1)}$$

where df is the number of degrees of freedom (given in Table C1) [Howe, 1981]. This table is valid only for overlapped estimates of the Allan variance.

To compute the confidence interval for $\hat{\sigma}_y(\tau = 1\text{ s}) = 10^{-12}$, for flicker frequency noise, N = 101, and $\tau_o = 0.5$ s (m = 2), we first find the number of degrees of freedom using Table C1:

$$df = \frac{5N^2}{4m(N+3m)} \qquad \text{(Eq. C2)}$$

$$= \frac{5(101^2)}{4(2)(101+3(2))} = 59.6.$$

For a 1σ (68%) confidence interval, the χ^2 values needed are $\chi^2(0.16)$ and $\chi^2(1\text{-}0.16)$. These values can be obtained from numerical tables of the chi-squared distribution function or from various computer programs. For this example $\chi^2(0.16) = 48.25$, and $\chi^2(0.84) = 69.73$.

Therefore

$$\chi^2(0.16) < \frac{59.6\hat{\sigma}_y^2}{\sigma_y^2} < \chi^2(0.84), \qquad \text{(Eq. C3)}$$

or

$$\frac{59.6\hat{\sigma}_y^2}{\chi^2(0.84)} < \sigma_y^2 < \frac{59.6\hat{\sigma}_y^2}{\chi^2(0.16)}, \qquad \text{(Eq. C4)}$$

$$0.85\hat{\sigma}_y^2 < \sigma_y^2 < 1.24\hat{\sigma}_y^2. \qquad \text{(Eq. C5)}$$

Other methods have been developed for computing the confidence interval of Mod $\sigma_y(\tau)$ [Walter, 1994; Greenhall, 1995]. Table C2 shows a comparison of confidence intervals for $\sigma_y(\tau)$ (no overlap and full overlap) and Mod $\sigma_y(\tau)$ for white PM, flicker PM, and white FM noise processes [Walls, 1995]. As shown in Table C2, *overlapped* estimates improve the confidence intervals for specific values of M and m. Although $\sigma_y(\tau)$ usually provides a better estimate (confidence interval) than Mod $\sigma_y(\tau)$ for fractional fluctuations (see Table C2 for white PM and flicker PM), the confidence interval for absolute fluctuations is approximately similar. The reason is that the value of Mod $\sigma_y(\tau)$ is typically much smaller than $\sigma_y(\tau)$ for white and flicker PM noise processes; therefore Mod $\sigma_y(\tau)$ provides a better estimate of stability.

Table C1. Empirical equations for the number of degrees of freedom of the Allan Variance estimate. From [Howe, 1981].

Noise Type	Degrees of Freedom
white phase	$\frac{(N+1)(N-2m)}{2(N-m)}$
flicker phase	$\exp\left[\ln\left(\frac{N-1}{2m}\right)\ln\left(\frac{(2m+1)(N-1)}{4}\right)\right]^{1/2}$
white frequency	$\left[\frac{3(N-1)}{2m}-\frac{2(N-2)}{N}\right]\frac{4m^2}{4m^2+5}$
flicker frequency	$\frac{2(N-2)^2}{2.3N-4.9}$ for $m=1$ $\frac{5N^2}{4m(N+3m)}$ for $m\geq 2$
random-walk frequency	$\frac{N-2}{m}\frac{(N-1)^2-3m(N-1)+4m^2}{(N-3)^2}$

Table C2. Confidence intervals for $\sigma_y(\tau)$ (no overlap and full overlap) and Mod $\sigma_y(\tau)$. From [Walls, 1995].

	No Overlap ± for 68% $\sigma_y(\tau)$	Full Overlap - for 68% $\sigma_y(\tau)$	Full Overlap + for 68% $\sigma_y(\tau)$	Full Overlap - for 68% Mod $\sigma_y(\tau)$	Full Overlap + for 68% Mod $\sigma_y(\tau)$
n=1025	White PM	White PM	White PM	White PM	White PM
m=2	4.4%	2.9%	3.2%	3.1%	3.4%
m=8	8.7%	2.9%	3.2%	5.2%	6.1%
m=32	17.4%	3.0%	3.4%	9.7%	14%
m=128	34.9%	3.1%	3.6%	18%	41%
n=1025	Flicker PM	Flicker PM	Flicker PM	Flicker PM	Flicker PM
m=2	4.4%	2.9%	3.1%	3.0%	3.3%
m=8	8.7%	3.6%	4.0%	5.7%	6.8%
m=32	17.4%	5.2%	6.1%	11%	16%
m=128	34.9%	8.4%	11%	20%	50%
n=1025	White FM	White FM	White FM	White FM	White FM
m=2	3.8%	2.8%	3.0%	3.0%	3.2%
m=8	7.7%	4.8%	5.6%	5.8%	7.0%
m=32	15.3%	8.8%	12%	11%	16%
m=128	30.6%	16%	32%	20%	51%

Appendix D

Bibliography

[1] Allan, D. W., "Statistics of Atomic Frequency Standards," Proc. IEEE, Vol. 54, pp. 221-230, Feb. 1966.

[2] Allan, D. W., et al., "Performance, Modeling, and Simulation of Some Cesium Beam Clocks," Proc. 27th Annual Symp. Freq. Control, pp. 334-346, 1973.

[3] Allan, D. W., "The Measurement of Frequency and Frequency Stability of Precision Oscillators," Proc. 6th Annual Precise Time and Time Interval Planning Meeting, pp. 109-142, Dec. 1974.

[4] Allan, D. W., and Daams, H., "Picosecond Time Difference Measurement System," Proc. 29th Annual Symp. Freq. Control, pp. 404-411, May 1975.

[5] Allan, D. W., and Hellwig, H. W., "Time Deviation and Time Prediction Error for Clock Specification, Characterization, and Application," IEEE Proc. Position Location and Navigation Symp., pp. 29-36, 1978.

[6] Allan, D. W., "Time and Frequency (Time Domain) Characterization, Estimation and Prediction of Precision Clocks and Oscillators," IEEE Trans. UFFC, Vol. 34, pp. 647-654, Nov. 1987.

[7] Atkinson, W. K., Fey, L., and Newman, J., "Spectrum Analysis of Extremely Low-Frequency Variations of Quartz Oscillators," Proc. IEEE, Vol. 51, pp. 379-380, Feb. 1963.

[8] Babitch, D., and Oliverio, J., "Phase Noise of Various Oscillators at Very Low Fourier Frequencies," Proc. 28th Annual Symp. Freq. Control, pp. 150-159, 1974.

[9] Barnes, J. A., "Models for the Interpretation of Frequency Stability Measurements," Nat. Bur. Stands. (U.S.) Tech. Note 683, Aug. 1976.

[10] Barnes, J. A., Jones, R. H., Tryon, P. V., and Allan, D.W., "Noise Models for Atomic Clocks," Proc. 14th Annual Precise Time and Time Interval Planning Meeting, pp. 295-307, Dec. 1982.

[11] Baugh, R. A., "Frequency Modulation Analysis with the Hadamard Variance," Proc. 25th Annual Symp. Freq. Control, pp. 222-225, 1971.

[12] Bernier, L.G., "Linear Prediction of the Non-Stationary Clock Error Function," Proc. 2nd European Frequency and Time Forum, Neuchâtel, 1988.

[13] Blackman, R. B., and Tukey, J. M., The Measurement of Power Spectra, Dover Publication, Inc., New York, NY, 1959.

[14] Blair, B. E., "Time and Frequency: Theory and Fundamentals," Nat. Bur. Stands. (U.S.) Monograph No. 140, US Government Printing Office, Washington, DC 20402, May 1974.

[15] Boileau, E., and Picinbonol, B., "Statistical Study of Phase Fluctuations and Oscillator Stability," IEEE Trans. Instrum. Meas., Vol. 25, pp. 66-75, March 1976.

[16] Brandenberger, H., Hadorn, F., Halford, D., and Shoaf, J. H., "High Quality Quartz Crystal Oscillators: Frequency-Domain and Time-Domain Stability," Proc. 25th Annual Symp. Freq. Control, pp. 226- 230, 1971.

[17] Chi, A. R., "The Mechanics of Translation of Frequency Stability Measures Between Frequency and Time- Domain Measurements," Proc. 9th Annual Precise Time and Time Interval Planning Meeting, Dec. 1977.

[18] Cutler, L. S., and Searle, C. L., "Some Aspects of the Theory and Measurement of Frequency Fluctuations in Frequency Standards," Proc. IEEE, Vol. 54, pp. 136-154, Feb. 1966.

[19] De Prins, J., and Cornelissen, G., Analyse Spectrale Discrete, Eurocon (Lausanne, Switzerland), Oct. 1971.

[20] De Prins, J., Descornet, G., Gorski, M., and Tamine, J., "Frequency-Domain Interpretation of Oscillator Phase Stability," IEEE Trans. Instrum. Meas., Vol. IM- 18, pp. 251-261, Dec. 1969.

[21] Fischer, M. C., "Frequency Stability Measurement Procedures," Proc. 8th Annual Precise Time and Time Interval Planning Meeting, pp. 575-618, Dec. 1976.

[22] Greenhall, C.A., "Initializing a Flicker Noise Generator," IEEE Trans. Instrum. Meas., Vol. IM-35, pp. 222-224 June 1986.

[23] Groslambert, J., Oliver, M., and Uebersfeld, J., "Spectral and Short-Term Stability Measurements," IEEE Trans. Instrum. Meas., Vol. IM-23, pp. 518-521, Dec. 1974.

[24] Halford, D., "A General Mechanical Model for |f| Spectral Density Noise with Special Reference to Flicker Noise 1/|f|," Proc. IEEE, Vol. 56, 251-258, March 1968.

[25] Howe, D.A., "Methods of Improving the Estimation of Long-Term Frequency Variance," Proc. European Frequency and Time Forum, 1997.

[26] IEEE, Special Issue on Time and Frequency, Proc. IEEE, Vol. 60, May 1972.

[27] IEEE-NASA, Proceedings of Symposium on Short-Term Frequency Stability, NASA Publication SP 80, 1964.

[28] Jones, R. H., and Tryon, P. V., "Estimating Time from Atomic Clocks," J. Res. Nat. Bur. Stands. (U.S.), Vol. 88, November 17-24, Jan.-Feb. 1983.

[29] Kartaschoff, P., "Computer Simulation of the Conventional Clock Model," IEEE Trans. Instrum. Meas., Vol. IM- 28, pp. 193-197, Sept. 1979.

[30] Kroupa, V. F., Frequency Stability: Fundamentals and Measurement, IEEE Press. IEEE Selected Reprint Series, prepared under the sponsorship the IEEE Instrumentation and Measurement Society, pp. 77-80, 1984.

[31] Lesage, P., and Audoin, C., "Correction to: Characterization of Frequency Stability: Uncertainty due to Finite Number of Measurements," IEEE Trans. Insrum., Meas., Vol. IM-23, pp. 103, March 1974.

[32] Lesage, P., and Audoin, F., "A Time Domain Method for Measurement of the Spectral Density of Frequency Fluctuations at Low Fourier Frequencies," Proc. 29th Annual Symp. Freq. Control, pp. 394 - 403, May 1975.

[33] Lesage, P., and Audoin, C., "Correction to: Characterization of Frequency Stability: Uncertainty due to the Finite Number of Measurements," IEEE Trans. Instrum. Meas., Vol. IM-25, pp. 270, Sept. 1976.

[34] Lesage, P., and Ayi, T., "Characterization of Frequency Stability: Analysis of Modified Allan Variance and Properties of its Estimate," IEEE Trans. Instrum. Meas., Vol. IM-33, pp. 332-336, Dec. 1984.

[35] Lindsey, W. C., and Chie, C. M., "Theory of Oscillator Instability Based Upon Structure Functions," Proc. IEEE, Vol. 64, pp. 1662-1666, Dec. 1976.

[36] Mandelbrot, B., "Some Noises with 1/f Spectrum; a Bridge between Direct Current and White Noise," IEEE Trans. Information Theory, Vol. IT-13, pp. 289-298, April 1967.

[37] Meyer, D. G., "A Test Set for the Accurate Measurements of Phase Noise on High-Quality Signal Sources," IEEE Trans. Instrum. Meas., Vol. IM-19, pp. 215-227, Nov. 1970.

[38] Percival, D. B., "A Heuristic Model of Long-Term Atomic Clock Behavior," Proc. 30th Annual Symp. Freq. Control, June 1976.

[39] Ricci, D. W., and Peregrino, L., "Phase Noise Measurement Using a High Resolution Counter with on-line Data Processing," Proc. 30th Annual Symp. Freq. Control, June 1976.

[40] Riley, W. J., "A Test Suite for the Calculation of Time Domain Frequency Stability," Proc. 1995 IEEE Int'l. Freq. Control Symp., pp. 360-366, June 1995.

[41] Riley, W. J., "Addendum to a Test Suite for the Calculation of Time Domain Frequency Stability," Proc. 1996 IEEE Int'l Freq. Control Symp., pp. 880-882, June 1996.

[42] Rutman, J., "Comment on Characterization of Frequency Stability," IEEE Trans. Instrum. Meas., Vol. IM- 21, pp. 85, Feb. 1972.

[43] Rutman, J., "Characterization of Frequency Stability: A Transfer Function Approach and its Application to Measurements via Filtering of phase Noise, IEEE Trans. Instrum. Meas., Vol. IM-23, pp. 40-48, March 1974.

[44] Rutman, J., "Characterization of Phase and Frequency Instabilities in Precision Frequency Sources: Fifteen Years of Progress," Proc. IEEE, Vol. 66, pp. 1048-1075, Sept. 1978.

[45] Rutman, J., and Sauvage, G., "Measurement of Frequency Stability in the Time and Frequency Domains via Filtering of Phase Noise," IEEE Trans. Instrum. Meas., Vol. IM-23, pp. 515-518, Dec. 1974.

[46] Rutman, J., and Uebersfeld, J., "A Model for Flicker Frequency Noise of Oscillators," Proc. IEEE, Vol. 60, pp. 233-235, Feb. 1972.

[47] Sauvage, G., and Rutman, J., "Analyse Spectrale du bruit de Frequence des Oscillateurs par la Variance de Hadamard," Ann. des Telecom., Vol. 28, pp. 304-314, July-Aug 1973.

[48] Vanier, J., and Tetu, M., "Time Domain Measurement of Frequency Stability: A Tutorial Approach," Proc. 10th Annual Precise Time and Time Interval Planning Meeting, pp. 247-291, 1978.

[49] Vessot, R., "Frequency and Time Standards," Methods of Experimental Physics, Academic Press, New York, pp. 198-227, 1976.

[50] Vessot, R., Mueller, L., and Vanier, J., "The Specification of Oscillator Characteristics from Measurements made in the Frequency Domain," Proc. IEEE, Vol. 54, pp. 199-207 Feb. 1966.

[51] Von Neumann, J., Kent, R. H., Bellinson, H. R., and Hart, B. I., "The Mean Square Successive Difference," Ann. Math. Stat., Vol. 12, pp. 153-162, 1941.

[52] Walls, F.L., and Allan, D.W., "Measurements of Frequency Stability," Proc. IEEE, Special Issue on Radio Measurements and Standards, Vol. 74, No. 1, pp. 162- 168, Jan. 1986.

[53] Walls, F. L., Stein, S. R., Gray, J. E., and Glaze, D. J., "Design Considerations in State-of-the-Art Signal Processing and Phase Noise Measurement Systems," Proc. 30th Annual Symp. Freq. Control, June 1976.

[54] Walls, F.L., "Measurement Techniques (Metrology)," in ITU-R Handbook on Selection and Use of Precise Frequency and Time Systems, Chapter 4, in preparation.

[55] Winkler, G. M. R., A Brief Review of Frequency Stability Measures," Proc. 8th Annual Precise Time and Time Interval Planning Meeting, pp. 489-528, U.S. Naval Research Laboratory, Washington, D.C., Dec. 1976.

[56] Winkler, G. M. R., Hall, R. G., and Percival, D. B., "The U.S. Naval Observatory Clock Time Reference and the Performance of a Sample of Atomic Clocks," Metrologia, Vol. 6, pp. 126-134, Oct. 1970.

[57] Yoshimura, K., "Characterization of Frequency Stability: Uncertainty due to the Autocorrelation of the Frequency Fluctuations," IEEE Trans. Instrum. Meas., IM-27, pp. 1-7, March 1978.

NOTE: Appendix D does not repeat the references given in Section 7 of the main text.

Appendix E

Definitions

The standard definitions given in the main body of the text are repeated here in narrative form. If ambiguities are created between the narrative definition given here and the mathematical equation given in the text, the text has priority.

Frequency Deviation y(t): instantaneous, normalized frequency departure from a nominal frequency.

Phase Deviation $\phi(t)$: instantaneous phase departure from a nominal phase.

Time Deviation x(t): instantaneous time departure from a nominal time.

Frequency Instability $S_y(f)$: one-sided spectral density of the frequency deviation.

Phase Instability $S_\phi(f)$: one-sided spectral density of the phase deviation.

Time Instability $S_x(f)$: one-sided spectral density of the time deviation.

Two-Sample Variance, $\sigma_y^2(\tau)$, also called the Allan variance: time average over the sum of the squares of the differences between successive readings of the frequency departure sampled over the sampling time τ, under the assumption that there is no dead time between the frequency departure samples.

Two-Sample Deviation, $\sigma_y(\tau)$, also called the square root of the Allan variance: this is the square root of the two-sample variance as defined above.

Time Interval Error TIE: the variation of the time difference between a real clock and an ideal uniform time scale following a time period t after perfect synchronization.

Confidence Limit I_α: the uncertainty associated with the estimate of the two-sample deviation from a finite data set with a finite number of measurements. Typically, I_α refers to a 1σ (68%) confidence level or error bar. However, other confidence intervals, in particular larger ones, are frequently used.