Analysis of Frequency Biases and Noise in Sampled Digital Frequency Servos for Primary Frequency Standards

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Abstract— **Unlike previous frequency standards at NIST, the main frequency servo of NIST-7, the U. S. primary frequency standard, does not lock a local oscillator to the atomic resonance. Instead it compares the frequencies of the atomic resonance and a reference oscillator. This paper presents an analysis of biases and noise in this digital servo. We show that a slight modification of the servo permits sensitive measurements for the presence of frequency biases.**

I. Introduction

NIST-7, the U. S. primary frequency standard, contributes to the computation of TAI by the Bureau International des Poids et Measures (BIPM), supports the NIST AT1 time scale, and serves as the U. S. primary frequency standard for calibration. NIST-7 is not operated as a clock. Instead, the frequency of a stable reference oscillator is periodically measured with respect to the primary standard with known biases removed. This reference oscillator serves as a transfer standard for conveying frequency information to the BIPM via GPS common view. NIST-7 also contributes to AT1 by providing information on frequency drift of the reference oscillator. Secondary frequency standards are calibrated at NIST by comparison with this stable reference oscillator. Each of these roles depends upon the accurate comparison of the frequency of a stable reference oscillator with the frequency of the cesium atomic resonance that defines the duration of the second.

II. Digital Frequency Servo

Figure 1 is a block diagram of the digital servo that compares the reference oscillator to the atomic resonance. The digital servo performs slow square-wave frequency modulation of a microwave synthesizer using a computer-controlled, direct-digital synthesizer

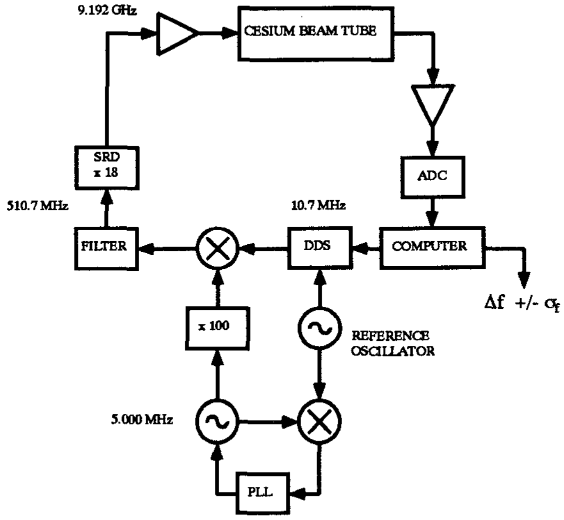

Fig. 1. Block Diagram of the Digital Frequency Servo. SRD=step recovery diode, DDS=direct digital synthesizer, PLL=phase locked loop, ADC=analog to digital converter.

(DDS). The resulting fluorescence signal from the optically pumped cesium beam is digitized or sampled by an analog-to-digital converter (ADC) and then demodulated by the computer. Details of this process can be found in [1]. This error signal drives a software controller which steers the DDS center frequency to force the error signal to zero. The frequency steering corrections are stored by the computer for later calculation of the mean frequency difference between the reference oscillator and the atomic resonance. Thus, the output of NIST-7 is not an electrical signal but a table of numbers.

For the analysis of the digital servo, we require that the stability of the frequency measurements be limited by the noise from the primary standard, not by that of the reference oscillator. Figure 2 illustrates the stability of NIST-7, the reference oscillator, and

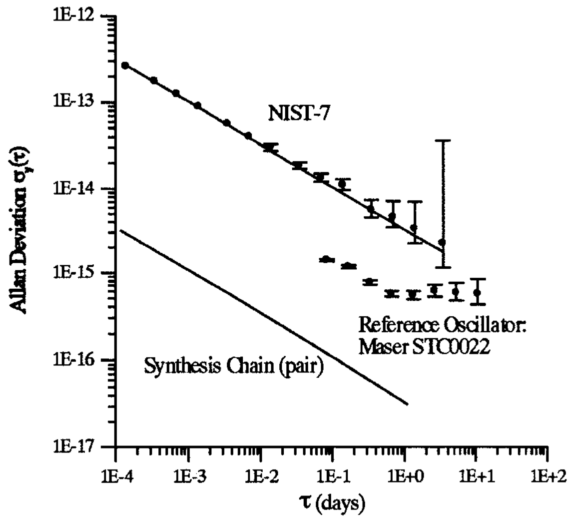

Fig. 2. Frequency Stability of NIST-7 and Reference Oscillator.

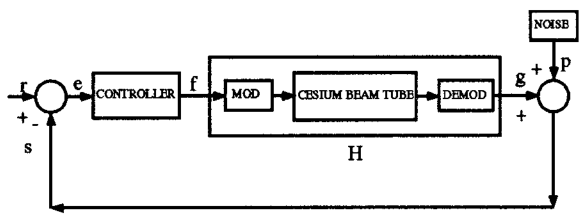

Fig. 3. Signal-Flow Diagram of the Digital Servo.

the microwave synthesis chain. The upper trace illustrates the frequency stability of NIST-7 relative to a hydrogen maser when either digital or traditional analog frequency servos are used. The middle data points show the frequency stability of our reference oscillator (a hydrogen maser).[2] The stability of the microwave synthesizer was determined by measuring the relative stablity of two similar synthesizers, driven by the same reference oscillator.[3] These data indicate that for averaging times of up to several days, the stability of the frequency comparison is indeed limited by the stability of the primary standard. To ensure that this is true for any particular frequency comparison, we continuously compare the frequency of the reference oscillator to the frequencies of several other devices with similar performance.

III. Software Controller

Figure 3 is a signal-flow diagram for our model of this digital frequency servo. The slow square-wave modulator (MOD), the cesium beam tube, and the demodulator (DEMOD) are modeled by a single transfer function H which produces the signal g. All of the noise in the system is modeled by the random process p with zero mean and white power spectral density. This model is used to analyze the servo, but it also accurately reflects the noise observed in NIST-7. Non-ideal noise sources, such as spurious coupling of the modulation waveform, are discussed in section IV. The noise p is added to g to give the signal s. In order to determine line center, the servo set-point r is zero. Thus the error signal e is simply $-s$. The output of the servo is the time-series f.

For the operation of a primary frequency standard, we desire that the controller satisfy three performance citeria:

- The mean $\bar{f}$ must be unbiased.
- There should be no excess noise on f.
- The statistical uncertainty of f must be unbiased.

Below, we show that a simple integral controller satisfies these criteria.

A. Mean Frequency

Let us assume that H has the form

$$H\{f(n)\} = \mu(f(n) - f_0). \tag{1}$$

This represents (for small detunings) a symmetric resonance centered at f_0 with slope $\pm\mu/2$ at $f_0 \mp f_m$, where f_m is the modulation amplitude. The output of an integral controller is the sum of the current error and all the previous errors

$$\begin{aligned} f(n) &= k[e(n) + e(n-1) + e(n-2) + \cdots] \quad (2) \\ &= f(n-1) + ke(n), \quad (3) \end{aligned}$$

where k is the integrator gain. From figure 3 and equation 1 we can write

$$f(n) = f(n-1)[1 - \mu k] + \mu k f_0 - kp(n). \tag{4}$$

The mean frequency produced by this servo is

$$\begin{aligned} \bar{f} &= \langle f(n) \rangle \\ &= \langle f(n-1)[1 - \mu k] + \mu k f_0 - kp(n) \rangle \\ &= f_0. \quad (5) \end{aligned}$$

Thus the servo satisfies the first criterion.

B. Frequency Stability

The ideal variance of the servo output is determined solely by the resonance lineshape and the system noise:

$$\sigma_f^2 = \frac{\sigma_p^2}{\mu^2}, \tag{6}$$

where σ_p^2 is the variance of the noise p. From equation 4 the variance of f is

$$\begin{aligned} \sigma_f^2 &= \langle f^2(n) \rangle - \langle f(n) \rangle^2 \\ &= \frac{k^2 \sigma_p^2}{k\mu(2 - k\mu)}. \quad (7) \end{aligned}$$

Solving for the integrator gain gives

$$k = \frac{2\mu}{(\sigma_p^2/\sigma_f^2) + \mu^2}. \tag{8}$$

Inserting equation 6 for ideal noise performance into equation 8 gives an expression for the ideal gain

$$K = \frac{1}{\mu}. \tag{9}$$

At ideal integrator gain the controller produces a time series f that reflects the linewidth and signal-to-noise ratio of the frequency standard. From the recursion relation 4, the ideal gain fully corrects for the frequency error from the previous iteration.

C. Statistical Uncertainty

An estimate of the mean of f

$$\hat{f} = \frac{1}{N}\sum_{m=1}^{N} f(m) \tag{10}$$

has variance

$$\sigma_{\hat{f}}^2 = \frac{\sigma_f^2}{N} \tag{11}$$

only if the autocorrelation of f satisfies

$$R_{ff}(m) = \begin{cases} \sigma_f^2 & \text{for } m = 0 \\ 0 & \text{for } m > 0. \end{cases} \tag{12}$$

For the integral controller, the autocorrelation is

$$\begin{aligned} R_{ff}(m) &= \langle f(n)f(n+m)\rangle \text{, for } m > 0 \\ &= (1 - k\mu)^m \sigma_f^2. \end{aligned} \tag{13}$$

For the ideal gain $k = K$, the autocorrelation vanishes for $m > 0$. Thus equation 11 is a valid expression for the uncertainty of $\hat{f}$ at ideal integrator gain. If $k \neq K$, then equation 11 is not strictly true and the estimate of $\sigma_{\hat{f}}^2$ is biased. To determine the magnitude of the bias, we performed a numerical simulation of the digital servo. The results are shown in figure 4. In order to keep the bias in the uncertainty of $\hat{f}$ below 10% the error in the measurement of K should not exceed 5%.

IV. Measured Frequency Biases

A frequency bias can occur when there is inadvertent coupling of the modulation waveform to the demodulator.[1] For example, this might occur in an optically pumped standard if there is amplitude modulation of the laser that is synchronous with the demodulator. The digital servo can be used to measure biases of this type. The first step is a slight modification

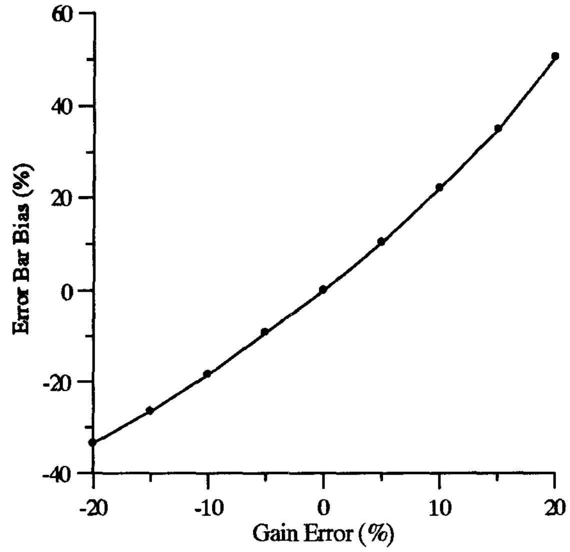

Fig. 4. Bias in Uncertainty vs. Gain Error.

of the software controller. The integral controller described above is replaced by a proportional controller that yields

$$f'(n) = Ae(n). \tag{14}$$

Next, the servo loop is interrupted in a way that both removes the frequency dependence of the error signal and emphasizes the bias of interest. For example, to measure laser amplitude modulation the atomic beam is blocked. Laser light is then scattered into the photodetector using frosted glass to reproduce the nominal signal level. The frequency bias that would occur under normal operation is calculated from

$$b = \frac{\hat{f}'}{\mu A}. \tag{15}$$

This technique has been used to verify that the fractional frequency bias in NIST-7 due to correlated amplitude modulation of the laser is $(0 \pm 2) \cdot 10^{-15}$. We stress that it is important to make as few changes as possible to the hardware and software during these bias measurements so that the results may be applied to the normal operation of the standard.

The digital demodulator acquires data from the primary standard by sampling the atomic fluorescence using an ADC. For the present configuration of NIST-7, to achieve a bias uncertainty of no more than $1 \cdot 10^{-15}$ requires an uncertainty in the error signal of no more than $4 \cdot 10^{-4}$ least significant bits (LSB). We verified our demodulator's ability to measure signals with amplitudes much less than 1 LSB by injecting the attenuated square-wave modulation waveform directly into the ADC. It was also necessary to add broadband

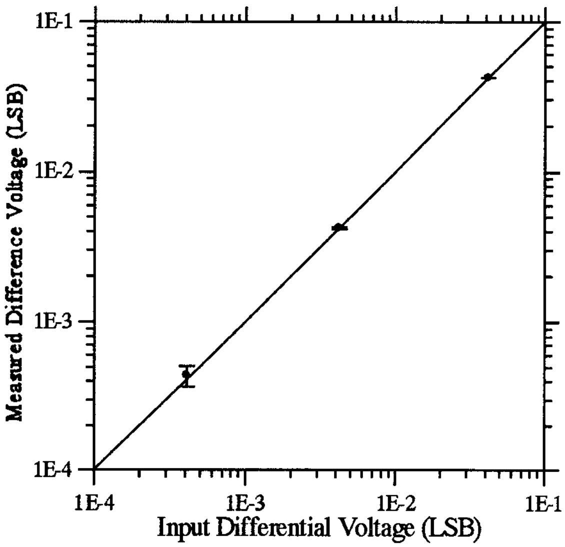

Fig. 5. Measurement of Signals Smaller than 1 LSB.

white noise with an amplitude of 1 LSBrms in order to overcome the quantization error of the ADC.[4] The results of several measurements are shown in figure 5. The solid line has a slope of 1 LSB/LSB. These measurements prove that we are able to detect extremely weak spurious coupling of the modulation waveform to the signal path.

V. Conclusion

We have developed a slow square-wave digital servo that measures frequency with an unbiased mean and variance. The servo is also useful for detecting systematic biases due to unintentional coupling of the modulation waveform to the signal path. For NIST-7 the measured frequency bias due to one such coupling is $(0 \pm 2) \cdot 10^{-15}$.

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