

VARIANCE ANALYSIS OF UNEVENLY SPACED TIME SERIES DATA

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Abstract

We have investigated the effect of uneven data spacing on the computation of $\sigma_x(\tau)$. Evenly spaced simulated data sets were generated for noise processes ranging from white PM to random walk FM. $\sigma_x(\tau)$ was then calculated for each noise type. Data were subsequently removed from each simulated data set using typical TWSTFT data patterns to create two unevenly spaced sets with average intervals of 2.8 and 3.6 days. $\sigma_x(\tau)$ was then calculated for each sparse data set using two different approaches. First, the missing data points were replaced by linear interpolation and $\sigma_x(\tau)$ calculated from this now full data set. The second approach ignored the fact that the data were unevenly spaced and calculated $\sigma_x(\tau)$ as if the data were equally spaced with average spacing of 2.8 or 3.6 days. Both approaches have advantages and disadvantages, and techniques are presented for correcting errors caused by uneven data spacing in typical TWSTFT data sets.

INTRODUCTION

Data points obtained from an experiment are often not evenly spaced. In this paper, we examine the application of $\sigma_x(\tau) = 3^{-1/2}\tau(\text{mod}\sigma_y(\tau))^{[1]}$ to the unevenly spaced time-series data obtained from two-way satellite time and frequency transfer (TWSTFT). We do so by using $\sigma_x(\tau)$ with both evenly and unevenly spaced simulated data of known power-law noise type and magnitude. The noise types examined are white phase modulation (WHPM), flicker phase modulation (FLPM), white frequency modulation (WHFM), flicker frequency modulation (FLFM), and random walk frequency modulation (RWFM)^[2].

Vernotte *et al.*^[3] studied the analysis of noise and drift in unevenly spaced pulsar data. However, the data obtained from pulsar studies are much more sparse in time, with only about 2% of the possible data available. In TWSTFT, the task is less daunting: time transfers are typically measured on Monday, Wednesday, and Friday, so, in a perfect world, we would have a data density of 3 data points present out of a possible 7.

This paper is not intended to be a rigorous treatment of how to calculate $\sigma_x(\tau)$ in all possible cases of unevenly spaced data. Rather, our purpose is to suggest methods and corrections which may be applied to data such as those produced by TWSTFT in order to obtain a more accurate assessment of the underlying time stability and noise type.

The National Institute of Standards and Technology (NIST) regularly performs time transfers with several laboratories in North America and Europe. Two of these laboratories are the United States Naval Observatory (USNO) in Washington, D.C. and the Van Swinden Laboratories (VSL) in Delft, the Netherlands. Typical data sets covering a 384-day period were chosen from the NIST-USNO and NIST-VSL time transfers to be used as templates.

METHOD OF EVALUATION

We evaluated the use of $\sigma_x(\tau)$ with unevenly spaced data having the five different power-law noise types: WHPM, FLPM, WHFM, FLFM, and RWFM. Ten independent data files were generated for each noise type. The WHPM, WHFM, and RWFM files were generated using a random-number generator and integration. The FLPM and FLFM files were generated according to the algorithm of Kasdin and Walter^[4]. All 10 data files of each noise type had 384 evenly spaced data points spaced one day apart. In the next step, we removed data points from each file so that the remaining data points aligned with the data points obtained from NIST-USNO or NIST-VSL TWSTFT. This produced files containing 137 or 108 unevenly spaced data points, respectively. The missing data points were then filled in by linear interpolation between the remaining data points. After this last step, there are once again 384 evenly spaced data points. Therefore, for each simulated data file of each noise type, we finally had five data files:

- File Type 1: the originally generated 384 evenly spaced data points with known noise type and magnitude.
- File Type 2: a data file of 137 data points spaced as in the NIST-USNO time transfers. This file is obtained by removing the appropriate data points from File 1. The average spacing (see below) is 2.816 days.
- File Type 3: File 2 with the missing data points filled in via linear interpolation.
- File Type 4: a data file of 108 data points spaced as in the NIST-VSL time transfers. This file, like File 2, is obtained by removing points from File 1. The average spacing (see below) is 3.579 days.
- File Type 5: File 4 with the missing data points filled in by linear interpolation.

Having created all 50 files for a given noise type, we then performed a $\sigma_x(\tau)$ analysis of each file. For the data files with even spacing (File Types 1, 3, and 5 above) we computed $\sigma_x(m\tau_{0,even})$ in the usual fashion^[1], where $m = 1, 2, 4, 8, 16, 32, 64, 128$ and $\tau_{0,even} = 1$ day. For the files with unevenly spaced data (File Types 2 and 4) we computed $\sigma_x(\tau)$ by treating the adjacent data points as if they were evenly spaced, with $\tau_{0,avg}$ calculated as follows:

$$\tau_{0,avg} = (MJD_{last} - MJD_{first}) / (N - 1) \quad (1)$$

where MJD_{first} and MJD_{last} are the time tags for the first and last data points, and N is the number of data points. For File Type 2, $\tau_{0,avg} = 2.816$ days, and for File Type 4, $\tau_{0,avg} = 3.579$ days. In both of these latter cases, we computed $\sigma_x(n\tau_{0,avg})$ for $n = 1, 2, 4, 8, 16, \text{ and } 32$. Having obtained $\sigma_x(\tau)$ vs τ for all 50 files, we then computed the average values of $\sigma_x(\tau)$ for each file type. Therefore, for each power-law noise type, we finally have five plots of $\sigma_x(\tau)$ vs

τ :

1. Average $\sigma_x(\tau) = 1, 2, 4, 8, 16, 32, 64,$ and 128 days) for File Type 1, that is, the files with known noise type. This plot shows the “correct” values for $\sigma_x(\tau)$.
2. Average $\sigma_x(\tau) = 2.816, 5.632, 11.264, 22.528, 45.056,$ and 90.112 days) for File Type 2. This represents the results we obtain by using unevenly spaced data with the NIST-USNO distribution.
3. Average $\sigma_x(\tau) = 1, 2, 4, 8, 16, 32, 64,$ and 128 days) for File Type 3. This represents the results we obtain by taking unevenly spaced data with the NIST-USNO distribution, performing linear interpolation to make an evenly spaced data file, and then performing the $\sigma_x(\tau)$ analysis.
4. Average $\sigma_x(\tau) = 3.579, 7.158, 14.316, 28.632, 57.264,$ and 114.528 days) for File Type 4. This represents the results we obtain by using unevenly spaced data with the NIST-VSL distribution.
5. Average $\sigma_x(\tau) = 1, 2, 4, 8, 16, 32, 64,$ and 128 days) for File Type 5. This represents the results we obtain by taking unevenly spaced data with the NIST-VSL distribution, performing linear interpolation to make an evenly spaced data file, and then performing the $\sigma_x(\tau)$ analysis.

Finally, for each average value of $\sigma_x(\tau)$ for File Types 2–5, we computed a “correction factor.” The correction factor is defined as

$$\text{correction factor}(\sigma_x(\tau)_{\text{File Type } j}) = \frac{\text{avg } \sigma_x(\tau)_{\text{File Type } 1}}{\text{avg } \sigma_x(\tau)_{\text{File Type } j}}. \quad (2)$$

In other words, multiplying the $\sigma_x(\tau)$ values obtained using File Type j by the correction factors for File Type j produces the correct value for $\sigma_x(\tau)$ as given by File Type 1. Because the τ values for File Types 2 and 4 do not match the τ values for File Type 1, various types of interpolation were used to obtain the correction factors for these two file types. The details of obtaining the correction factors for the different noise types and file types are discussed in the next section.

RESULTS

Figures 1–5 show the results obtained for the noise types WHPM, FLPM, WHFM, FLFM, and RWFM. Each of the points shown corresponds to the mean of ten values. The standard deviation of each set of ten values was also computed, but, for visual clarity, error bars indicating 11 standard deviation are shown only on the File Type 1 (i.e., correct) values. Approximately the same size error bars should be applied to each of the file type curves.

Figure 1 shows the results obtained for white PM noise. There are several important points here. First of all, File Types 3 and 5 (interpolating unevenly spaced data to form evenly spaced

data) yield values of $\sigma_x(\tau)$ which are much too small when τ is less than the $\tau_{0,avg}$ of the corresponding unevenly spaced data set. On the other hand, File Types 2 and 4 (the unevenly spaced data) yield $\sigma_x(\tau)$ values which have the $-1/2$ slope appropriate to white PM^[1], but which are consistently too high. In fact, for $\tau \geq 8$ days, both of the methods used converge to yield approximately the same too-large values for $\sigma_x(\tau)$. For File Types 2 and 4, the white PM correction factor is in theory constant for all values of τ and can be expressed as:

$$\text{correction factor (WHPM)} = \left(\frac{\tau_{0,even}}{\tau_{0,avg}} \right)^{1/2} \quad (3)$$

This occurs because with WHPM noise each data point in the time series is independent of all others.

Figure 2 shows the flicker PM results. Once again, File Types 3 and 5 yield values of $\sigma_x(\tau)$ which are too small at short averaging times. Also, the lower- τ values of $\sigma_x(\tau)$ for File Types 2 and 4 are again too high. However, the results obtained from all file types converge toward the correct value as τ increases. Similar results are obtained for white FM (Figure 3) and flicker FM (Figure 4).

Figure 5 shows the RWFM results. Here, the use of interpolated data (File Types 3 and 5) provides virtually the same results as the originally generated data file (File Type 1) and the use of unevenly spaced data (File Types 2 and 4) provides values of $\sigma_x(\tau)$ which are too large at small τ . In fact, as we progress from the WHPM process to the low-frequency-dominated noise processes (e.g., RWFM)^[2], the use of linear interpolation to fill in missing data points becomes an increasingly better approximation of the truth. For lower values of τ , using the unevenly spaced data becomes an increasingly worse approximation of the truth. As we progress from FLPM to RWFM, the results obtained using all methods converge on the correct value as τ increases.

>From the results shown in Figures 1-5 we have computed correction factors. Table 1 shows the correction factors obtained from the file types (3 and 5) which have evenly spaced data. These correction factors were obtained by simply taking the ratio

$$\frac{\sigma_x(m\tau_{0,even})_{FileType1}}{\sigma_x(m\tau_{0,even})_{FileType3or5}}$$

Tables 2-3 show the correction factors obtained for the file types (2 and 4) with unevenly spaced data. Because the averaging times for the unevenly spaced files (e.g. 2.816, 5.632, ..., etc. days for File Type 2) do not match the averaging times for File Type 1 (1, 2, 4, ..., etc. days), we cannot simply take a ratio of two values to get the correction factor. Generally, interpolation of some sort is required. Note that the correction factors for WHPM in Tables 2 and 3 all fall within 10% of the values calculated from Equation (3).

DISCUSSION

There is, unfortunately, no way to apply these results blindly. The user will need to have an idea of what sort of noise types make sense in the context of his measurement. Initially, one should construct one $\log \sigma_x(\tau)$ vs $\log(\tau)$ plot using the original set of unevenly spaced data and one $\log(\sigma_x(\tau))$ vs. $\log(\tau)$ plot using a full data set formed by linear interpolation.

At medium-to-large averaging times (in our analysis, $\tau \geq 8$ days), almost all methods, in their uncorrected state, provide the correct slope for the $\log \sigma_x(\tau)$ vs $\log(\tau)$ plot. For WHPM, the unevenly spaced data give the correct slope at all values of τ . Thus, the user can determine which power-law noise process dominates at medium-to-long averaging times. (The exception to this rule occurs when RWFM predominates, and the unevenly spaced data are used to make the $\log \sigma_x(\tau)$ vs. $\log(\tau)$ plot. In this case, the slope of the plot is slow in converging to the correct $+3/2$ value.) The more difficult part arises when the value of m in $\tau = m\tau_0$ is small. It is here that we see the largest effects of not having an evenly spaced data set. In addition, in this regime the noise process which dominates a measurement often changes from one type to another.

If data are recorded on Monday, Wednesday, and Friday, it will be impossible to get a reliable estimate of $\sigma_x(\tau = 1 \text{ day})$ — that information simply is not available. We can, however, make a fair estimate of $\sigma_x(\tau = 2 \text{ days})$ in this case because Monday-Wednesday and Wednesday-Friday are each two-day intervals. To be completely safe, one could avoid stating values of $\sigma_x(\tau)$ for $\tau < \tau_{0,avg}$. Finally, in this analysis, the ratio of the data length (384 days) to $\tau_{0,avg}$ (2.816 and 3.579 days) was always greater than 100; therefore, it may not be appropriate to use these results with short, sparse data sets.

If there is only one, known, noise type present, then the correction factors shown in Tables 1-3 can be applied. Unless one has exactly the same average data spacing as we did, some interpolation may be needed in order to use the correction factors. Fortunately, the values of most of the correction factors are not strongly dependent on the average spacing for the range of spacing that was examined. If the noise type is not known, one could begin by deciding whether their results contain only measurement noise, or if there is a mixture of measurement noise and clock noise. Examples of the former are common-clock or closure TWSTFT experiments. An example of the latter is performing TWSTFT between two remotely located clocks. We examine each of these situations below.

MEASUREMENT NOISE

If the results contain only measurement noise, then the noise type will most likely be white PM or flicker PM. Fortunately, as Figure 1 shows, if WHPM is the dominant noise type, the $\log \sigma_x(\tau)$ vs $\log(\tau)$ plot for the unevenly spaced data will have a clear $-1/2$ slope and it will be obvious that the WHPM corrections should be applied. This method was used in Reference 5. Similarly, if the $\log \sigma_x(\tau)$ vs $\log(\tau)$ plot has zero slope at large τ (Figure 2), then apply the FLPM corrections. In this case it is important to be certain that the noise type at large τ has been correctly ascertained because, if the noise type is FLPM, the corrections which are applied at large τ are fairly small. If the noise type is WHPM, the corrections which are

applied at large τ are relatively large.

COMBINATION OF CLOCK NOISE AND MEASUREMENT NOISE

If the experiment measures clock behavior (or some other quantity which is characterized by a low-frequency-dominated noise type), then the situation becomes more complicated because the results will contain a mixture of noise types – the noise type associated with the measurement and the noise type(s) associated with the behavior of the clocks under study. We have evaluated various analysis techniques and have arrived at the following recommendations which combine ease of use with acceptable accuracy.

First, examine the $\sigma_x(\tau)$ plots for evidence of measurement noise (WHPM, FLPM). The simplest way to see if there is any measurement noise is to look at the $\sigma_x(\tau)$ plot of the interpolated data set in the region where τ is small to medium. As Figures 1-3 show, for WHPM, FLPM, and WHFM, the $\sigma_x(\tau)$ plot of the interpolated data will curve down as τ decreases to approach $\tau = 1$ day. In the case of FLFM, the $\sigma_x(\tau)$ plot of the interpolated data makes a straight line as τ decreases. In the case of RWFM, the $\sigma_x(\tau)$ plot curves up slightly as τ decreases. Therefore, if the curve is downward at small τ and if there is evidence of a flat transition area at medium τ , there is probably significant measurement noise present.

If there indeed is measurement noise mixed in with the long-term noise, we suggest the following procedure (hereafter called the “hybrid method”): compute $\tau_{0,avg}$ from the unevenly spaced data and then simply use the $\sigma_x(\tau)$ values obtained from the interpolated data for $\tau > \tau_{0,avg}$. Then, estimate $\sigma_x(m\tau_{0,even})$, where $m\tau_{0,even}$ is the largest integral multiple of $\tau_{0,even}$ that is less than $\tau_{0,avg}$, as follows:

1. Using the values of $\log \sigma_x(\tau = \tau_{0,avg})$ and $\log \sigma_x(\tau = 2\tau_{0,avg})$ obtained from the unevenly spaced data, perform a linear extrapolation to smaller τ to obtain an estimate for $\log \sigma_x(\tau = m\tau_{0,even})$ for the unevenly spaced data set.
2. Compute the average of $\log \sigma_x(\tau = m\tau_{0,even})$ obtained from Step 1 and $\log \sigma_x(\tau = m\tau_{0,even})$ obtained from the interpolated data set.
3. Use this average value as an estimate of the correct value of $\log \sigma_x(\tau = m\tau_{0,even})$.

For example, the NIST-USNO data have $\tau_{0,avg} = 2.816$ days. Therefore, to obtain values of $\sigma_x(4 \text{ days} \leq \tau \leq 128 \text{ days})$ we would use the $\sigma_x(\tau)$ values obtained from the interpolated data. To get an estimate of $\sigma_x(\tau = 2 \text{ days})$ we would use the three steps outlined above. Further examples of this process are presented below.

This technique works because, for typical clock noise types (WHFM, FLFM, RWFM), the uncorrected values obtained from the interpolated data set are a pretty good estimate of the true values for medium to long averaging times. For measurement noise types WHPM, FLPM, and WHFM, at small values of τ , taking the average of the logarithm of $\sigma_x(\tau)$ associated with the interpolated and the unevenly spaced data sets yields an acceptable estimate of the true value of $\sigma_x(\tau)$. If inspection of the $\sigma_x(\tau)$ plots reveals no hint of measurement noise (i.e., it appears that clock noise dominates even at small τ , then determine the noise type from the

large- τ values of $\sigma_x(\tau)$ and then apply the appropriate correction factors from Table 1 to the $\sigma_x(\tau)$ values obtained from the interpolated data set.

We now show three examples of the analysis of mixed noise types, ranging from situations in which the measurement noise dominates out to medium τ to situations in which the measurement noise is quickly overwhelmed by clock behavior. In Combination 1 (Figures 6a–6b), we see a case in which inspection of the initial $\sigma_x(\tau)$ plots (Figure 6a) reveals obvious signs of the presence of both measurement and clock noise. The average data spacing is 2.816 days. As Figure 6b shows, using the hybrid method provides very good estimates of the correct values of $\sigma_x(\tau)$: the largest error is only 10% of the true $\sigma_x(\tau)$. In addition, we do not need to know precisely what types of noise are present (in this case, WHPM and WHFM) in order to arrive at the final estimates for $\sigma_x(\tau)$. Finally, we do not attempt to obtain a value for $\tau = 1$ day.

In Combination 2, we again see signs of both measurement noise and clock noise in the initial $\sigma_x(\tau)$ plots (Figure 7a). The average data spacing for Combinations 2 and 3 (see below) is 3.008 days. As Figure 7b shows, the hybrid method again provides a good estimate of the correct values for this combination of WHPM and FLFM.

In Combination 3, it is difficult to tell if there is any measurement noise present. The $\sigma_x(\tau)$ plot of the interpolated data set exhibits a very faint downward curve as τ decreases toward 1 day, but other than that, it looks like FLFM (Figure 8a). We have used both the hybrid technique and the simple application of the FLFM corrections (Table 1). As Figure 8b shows, the FLFM corrections work marginally better. As it turns out, the true $\sigma_x(\tau)$ curve shows clear evidence of measurement noise (WHPM) only at $\tau = 1$ day – a time interval about which we can gain no information from the sparse ($\tau_{0,avg} = 3.008$ days) data set.

CONCLUSIONS

We have used two typical TWSTFT time series data sets to investigate the impact of unevenly spaced data on the calculation of $\sigma_x(\tau)$. We have analyzed simulated data sets that have had points removed to match the TWSTFT data patterns. $\sigma_x(\tau)$ was calculated from these sparse data sets using two techniques. One involves analyzing the sparse data as if they were evenly spaced with an average time interval, and the second uses interpolated data to recreate an evenly spaced data set. Correction factors for both approaches have been calculated for noise processes ranging from WHPM to RWFM. For all of the noise processes except WHPM, the values of $\sigma_x(\tau)$ calculated with either of the two approaches converge on the correct values at large τ . However, significant errors may be introduced for small τ . Finally, we suggest techniques for estimating correct values of $\sigma_x(\tau)$ in situations where the type of noise is unknown or where more than one noise type is present.

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REFERENCES

- [1] D.W. Allan, M.A. Weiss, and J.L. Jespersen 1991, "*A frequency-domain view of time-domain characterization of clocks and time and frequency distribution systems*, Proceedings of the 45th Annual Symposium on Frequency Control, 29-31 May 1991, Los Angeles, California, pp. 667-678.
- [2] D.W. Allan 1987, "*Time and frequency (time-domain) characterization, estimation, and prediction of precision clocks and oscillators*," *IEEE Trans. Ultrasonics, Ferroelectrics, and Frequency Control*, 1987, UFFC-34, 647-654.
- [3] F. Vernotte, G. Zalamasky, and E. Lantz 1994, "*Noise and drift analysis of non-equally spaced timing data*," Proceedings of the 25th Annual Precise Time and Time Interval (PTTI) Applications and Planning Meeting, 29 November-2 December 1993, pp. 379-388.
- [4] N.J. Kasdin, and T. Walter 1992, "*Discrete simulation of power law noise*," Proceedings of the 1992 IEEE Frequency Control Symposium, 27-29 May 1992, Hershey, Pennsylvania, pp. 274-283.
- [5] C. Hackman, S.R. Jefferts, and T. Parker 1995, "*Common-clock two-way satellite time transfer experiments*," Proceedings of the 1995 IEEE Frequency Control Symposium, 31 May-2 June 1995, San Francisco, California, pp. 275-281.

Table 1: Correction Factors for File Types 3 and 5.

τ (days)	WHIPM		FLPM		WHIFM		FLFM		RWFPM	
	USNO	VSL	USNO	VSL	USNO	VSL	USNO	VSL	USNO	VSL
1.0	4.02	4.74	3.28	4.02	2.71	3.24	1.94	2.15	1.22	1.23
2.0	1.56	1.83	1.51	1.85	1.51	1.80	1.38	1.53	1.16	1.18
4.0	0.88	0.93	1.07	1.33	1.16	1.32	1.13	1.24	1.07	1.11
8.0	0.67	0.64	0.98	1.01	1.05	1.13	1.04	1.08	1.02	1.06
16.0	0.60	0.53	0.98	0.96	1.01	1.02	1.01	1.02	1.01	1.02
32.0	0.61	0.57	0.95	0.94	1.00	1.00	1.00	1.01	1.00	1.00
64.0	0.58	0.53	0.91	0.93	1.00	1.00	1.00	1.00	1.00	1.00
128.0	0.50	0.49	1.02	1.08	1.00	1.00	1.00	1.00	1.00	1.00

Table 2: Correction Factors for File Type 2.

τ (days)	WHIPM	FLPM	WHIFM	FLFM	RWFPM
	2.816	0.60	0.67	0.80	0.64
4.0	0.59	0.71	0.87	0.76	0.27
5.632	0.59	0.81	0.92	0.86	0.41
8.0	0.58	0.85	0.95	0.94	0.57
11.264	0.58	0.90	0.97	0.99	0.71
16.0	0.57	0.94	0.99	1.01	0.84
22.528	0.57	0.97	1.00	1.01	0.92
32.0	0.56	0.94	1.00	1.01	0.97
45.056	0.56	0.89	1.02	1.01	1.00
64.0	0.55	0.88	1.04	1.02	1.00
90.1128	0.55	0.95	1.08	1.05	1.00

Table 3: Correction Factors for File Type 4.

τ (days)	WHIPM	FLPM	WHIFM	FLFM	RWFPM
	3.579	0.49	0.65	0.80	0.50
4.0	0.49	0.66	0.82	0.56	0.14
7.158	0.50	0.80	0.91	0.81	0.33
8.0	0.50	0.81	0.92	0.85	0.38
14.316	0.50	0.91	0.97	0.94	0.59
16.0	0.50	0.91	0.98	0.95	0.63
28.632	0.51	0.91	1.00	0.93	0.78
32.0	0.51	0.90	1.01	0.92	0.80
57.264	0.52	0.80	1.03	0.92	0.90
64.0	0.52	0.81	1.04	0.92	0.93
114.528	0.52	1.04	1.09	1.06	1.10

Average $\sigma_x(\tau)$ vs. τ for WHPM

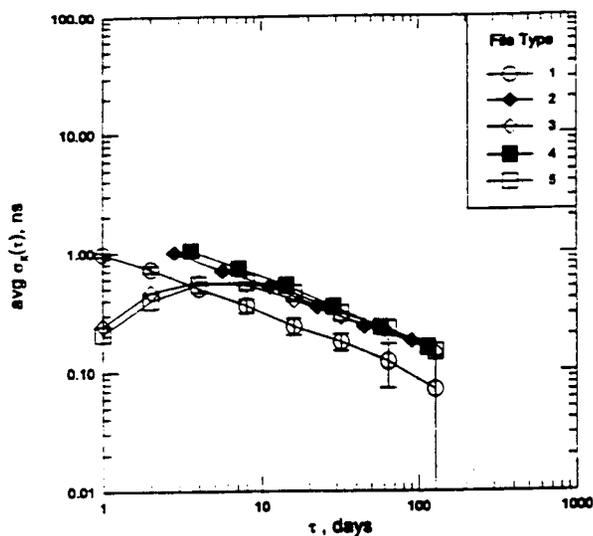


Figure 1.

The average values of $\sigma_x(\tau)$ obtained from simulated WHPM data. "File Type 1" indicates the correct values obtained from the original evenly spaced simulated data. "File Type 2" and "File Type 3" show the results obtained when some of the original data points are deleted, thus forming an average data spacing of 2.816 days, and then the remaining points analyzed two different ways. "File Type 4" and "File Type 5" indicate results obtained when data are decimated to produce an average data spacing of 3.579 days. For visual clarity, the error bars are not shown for File Types 2-5. However, the sizes of the missing error bars are approximately the same as those shown for File Type 1.

Average $\sigma_x(\tau)$ vs. τ for FLPM

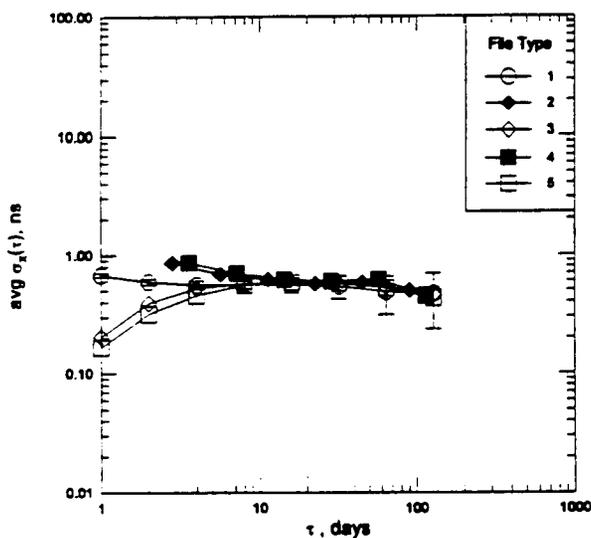


Figure 2.

The average values of $\sigma_x(\tau)$ obtained from simulated FLPM data.

Average $\sigma_x(\tau)$ vs. τ for WHFM

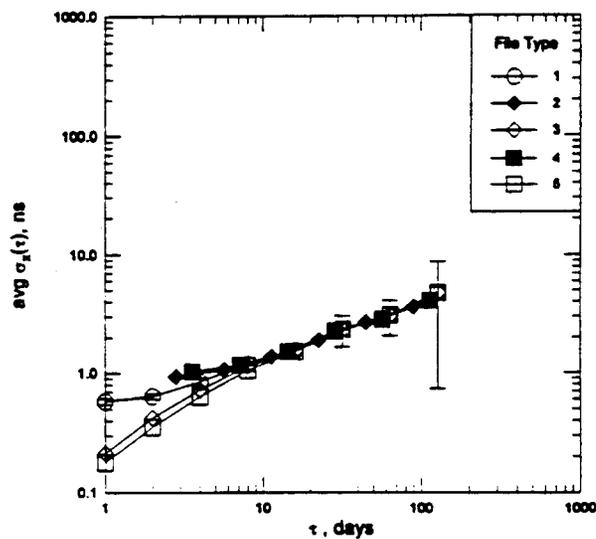


Figure 3.

The average values of $\sigma_x(\tau)$ obtained from simulated WHFM data.

Average $\sigma_x(\tau)$ vs. τ for FLFM

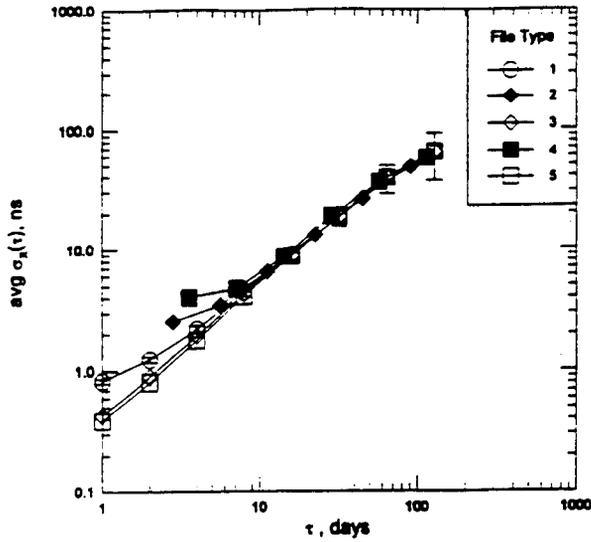


Figure 4.
The average values of $\sigma_x(\tau)$ obtained from simulated FLFM data.

Average $\sigma_x(\tau)$ vs. τ for RWFM

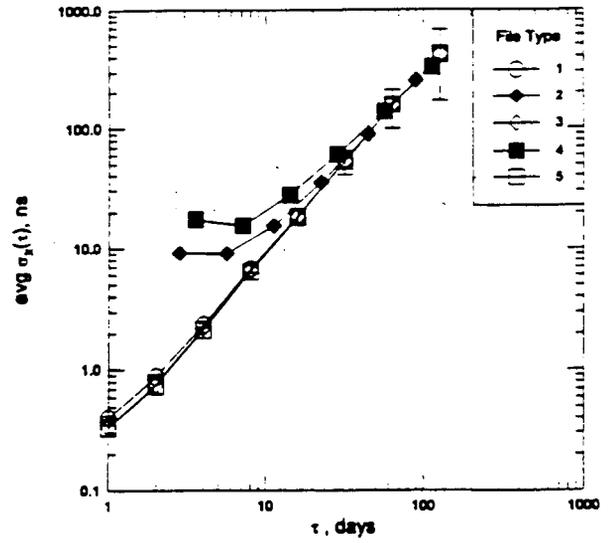


Figure 5.
The average values of $\sigma_x(\tau)$ obtained from simulated RWFM data.

COMBINATION 1

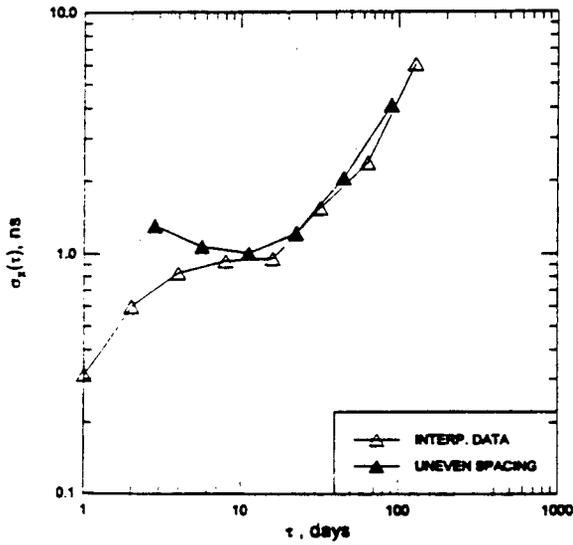


Figure 6a.
Uncorrected $\sigma_x(\tau)$ values obtained from a sparse data set with a mixture of WHPM and WHFM noise types.

COMBINATION 1

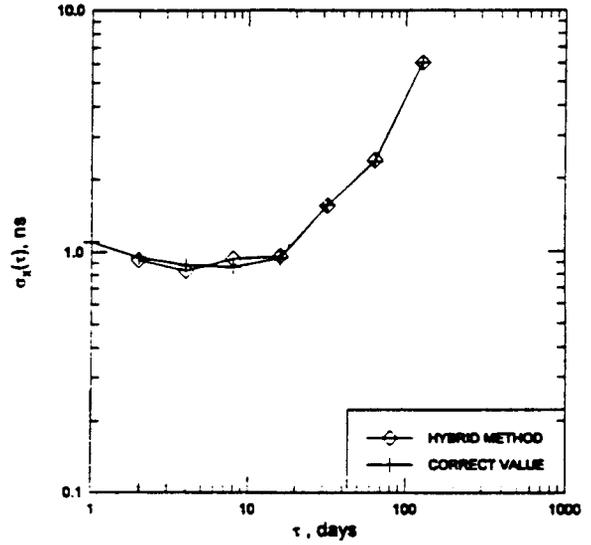


Figure 6b.
Corrected values of $\sigma_x(\tau)$ obtained using the "hybrid" method and the values obtained from the original, evenly spaced data set.

COMBINATION 2

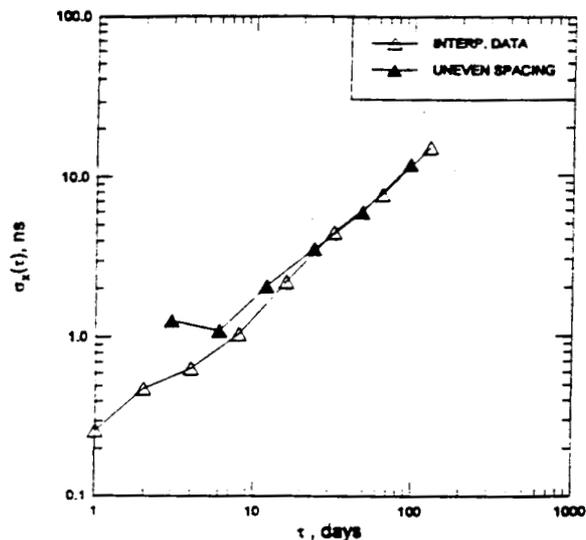


Figure 7a. Uncorrected $\sigma_x(\tau)$ values obtained from a sparse data set with a mixture of WHPM and FLFM noise types.

COMBINATION 2

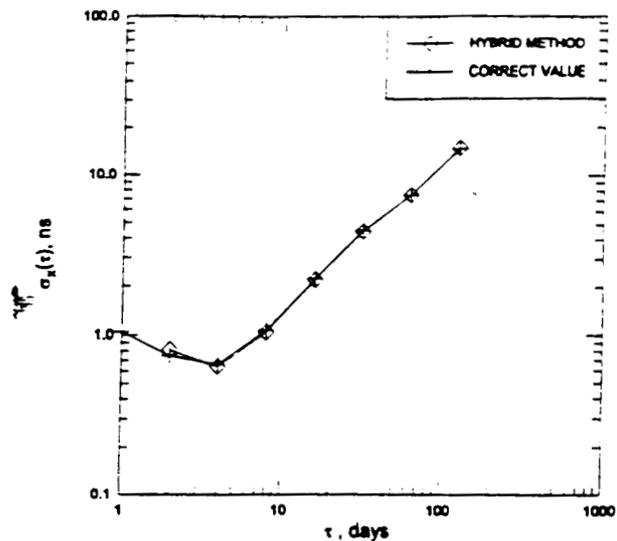


Figure 7b. Corrected values of $\sigma_x(\tau)$ obtained using the "hybrid" method and the values obtained from the original, evenly spaced data set.

COMBINATION 3

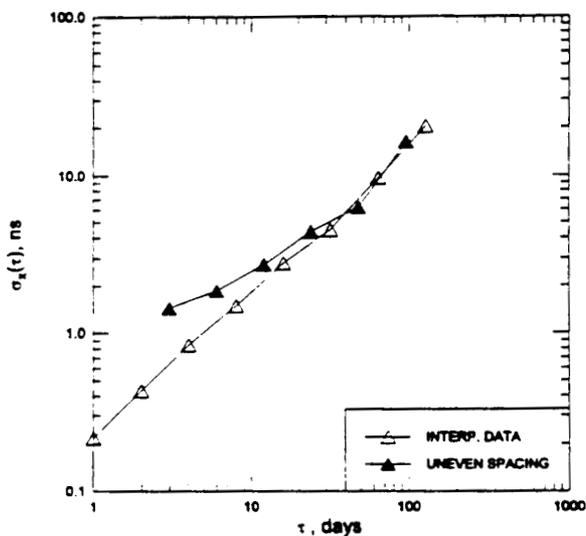


Figure 8a. Uncorrected $\sigma_x(\tau)$ values obtained from a sparse data set with a different mixture of WHPM and FLFM noise types.

COMBINATION 3

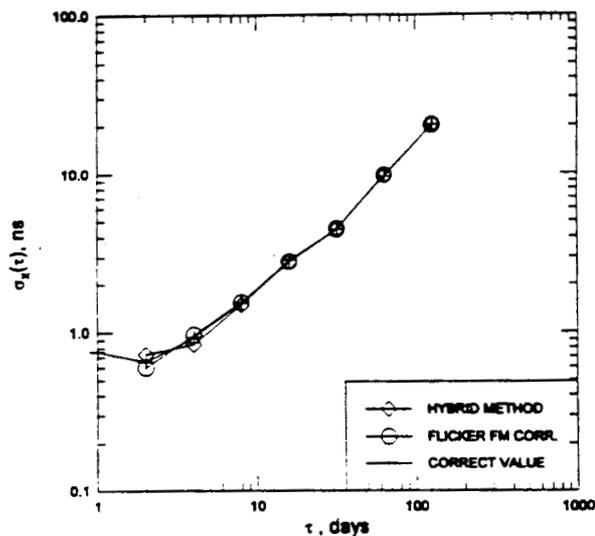


Figure 8b. Corrected values of $\sigma_x(\tau)$ obtained using the "hybrid" method, FLFM corrections only, and the values obtained from the original, evenly spaced data set.