IEEE TRANSACTIONS ON INSTRUMENTATION AND MEASUREMENT, VOL. 44, NO. 2, APRIL 1995

Minimizing the Time-Dilation Shift in Penning Trap Atomic Clocks

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Abstract—**If environmental perturbations are made negligible, the time-dilation shift is expected to give one of the largest systematic uncertainties in a stored ion clock. In general, this shift increases with the number of trapped ions. Fluctuations in the time dilation shift therefore could limit the frequency stability of an ion clock. We show that in a Penning trap, relativistic time dilation can be minimized if the laser-cooled ions are prepared in a special spheroidal state. In addition, a modest stabilization of the spheroid near the minimum-shift configuration can significantly reduce fluctuations in the time dilation. The results obtained for a single-species ion clock also provide a good approximation for a sympathetically cooled system.**

I. INTRODUCTION

ONE of the goals of the Time and Frequency Division at NIST-Boulder is to realize an ion frequency standard with absolute uncertainty better than 1 part in 10^{15}. In recent years, experiments at NIST with $^9\mathrm{Be}^+$ ions stored in a Penning trap have demonstrated that an RF oscillator can be steered by a nuclear spin-flip "clock" transition ($\sim$ 303 MHz) with $\sigma_y(\tau) \approx 3\times 10^{-12}/\sqrt{\tau}$ fractional frequency stability [1]–[4]. If collisional shifts [3], [4] are negligible, the expected performance of a stored ion clock using the Ramsey method [5] of interrogation is limited by the "projection-noise" [6] frequency instability

$$\sigma_y(\tau) = \frac{1}{2\pi\nu_o\sqrt{NT_R\tau}} \tag{1}$$

and the uncertainty in the time-dilation shift (second-order Doppler shift)

$$\frac{\Delta\nu}{\nu_o} = -\frac{1}{2}\frac{\langle v^2\rangle}{c^2}. \tag{2}$$

The frequency stability (1), depends on the clock transition frequency ν_o, the number of ions N, the Ramsey interrogation period T_R, and the averaging time $\tau \gg T_R$. The possibility of using "squeezed" atomic states to obtain higher frequency stability than (1) is being explored [7]. The time-dilation shift, (2), depends on the average squared velocity $\langle v^2\rangle$ of the ions. A systematic error of 5×10^{-15} due to time dilation has been determined for a stored $^9\mathrm{Be}^+$ ion clock [1]–[4]. In this paper, we give a detailed treatment of the time-dilation effects in Penning traps to identify the important factors for optimized performance [8].

II. LASER-COOLED, SINGLE-SPECIES ION CLOCK

The simplest stored-ion clock consists of one trapped ion, with charge q and mass M. In an ideal Penning trap, the ion is confined by a pure quadrupole electrostatic potential superimposed on a uniform magnetic field. The motions of the trapped ion consist of a harmonic oscillation (angular frequency ω_z) along the trap's symmetry axis, a high-frequency cyclotron orbit (ω_c') about the magnetic field, and a low-frequency magnetron orbit ($\omega_m = \omega_z^2/2\omega_c'$) about the trap z-axis that generates a $q\mathbf{v}\times\mathbf{B}$ force for radial confinement [9]. A suitable internal transition (e.g., between hyperfine levels) which is field-independent at a particular field B_o, can be used in a frequency standard. If the ion motions are cooled (or heated in the case of the magnetron orbit) to their ground states, the time-dilation shift in the selected "clock transition" is very small. The potentially high accuracy obtainable with a single ion ($N = 1$), however, exacts a cost in low signal-to-noise ratio and long averaging times.

The signal-to-noise ratio and frequency stability can be improved significantly by using large numbers of ions—see (1). However, the time-dilation shift also increases with the number of ions. This is because, in general, a larger ion "cloud" extends farther from the trap axis, and because, for stable trapping, an ion cloud or plasma [10], [11] must rotate (spin) about the trap's symmetry axis (taken as z-axis). This rotation frequency ω_r lies in the range

$$\omega_m < \omega_r < \omega_c' \tag{3}$$

where $\omega_c' = \omega_c - \omega_m$, and ω_c is the cyclotron frequency $\omega_c = qB_o/M$. The accompanying thermal motions of the ions can be reduced greatly by Doppler laser cooling (typically to < 1 K, with a cooling limit of $\sim$ 1 mK for $^9\mathrm{Be}^+$). We can assume that the time dilation due to thermal motions is negligible, and hereon study the $T = 0$ K limit. In a nearly ideal Penning trap [9], the laser-cooled ions form a uniform density spheroid bounded by

$$\left(\frac{x}{r_s}\right)^2 + \left(\frac{y}{r_s}\right)^2 + \left(\frac{z}{z_s}\right)^2 = 1 \tag{4}$$

with an aspect ratio $\alpha \equiv z_s/r_s$ which varies in a known way with the rotation frequency ω_r [10], [11]. When the $q\mathbf{v}\times\mathbf{B}$ radial restoring force is weak ($\omega_r \rightarrow \omega_m$) or when the centrifugal force is very strong ($\omega_r \rightarrow \omega_c'$), the spheroid is stretched into a thin circular disk ($\alpha \rightarrow 0$). Hence, near the upper and lower bounds of ω_r, the magnitude of the time-dilation shift ($\Delta\nu/\nu_o = -\omega_r^2 r_s^2/5c^2$) for the thin disk of ions

Manuscript received July 1, 1994; revised October 15, 1994. This work was supported in part by the Office of Naval Research.

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IEEE Log Number 9408707.

is very large. We now show that the magnitude of the time-dilation shift goes to a minimum at a particular (low) value of ω_r which brings the ions closer to the trap axis.

For a single ion species, the time-dilation shift due to the plasma rotation may be written as

$$\frac{\Delta\nu}{\nu_o} = -S_1 X^2 Y^2. \tag{5}$$

We use a dimensionless spheroid radius $X = r_s/b$ scaled by the radius b of a fictitious sphere enclosing N ions with the minimum cold fluid ($T = 0$ K) density $n_E = \epsilon_o M\omega_z^2/q^2$. This mass-independent radius b may be written as $b = \sqrt[3]{3N\tilde{r}c^2/\omega_z^2}$ where $\tilde{r} \equiv q^2/(4\pi\epsilon_o Mc^2)$. The dimensionless rotation frequency Y is given by $Y = (\omega_r/\omega_m)(\omega_c/\omega_c')$. The scaling factor $S_1 > 0$ is defined by

$$S_1 \equiv \frac{1}{20}\left[\frac{\omega_z}{\omega_c}\right]^2\left[3N\frac{\omega_z\tilde{r}}{c}\right]^{2/3}. \tag{6}$$

The radius and rotation frequency of the spheroidal ion cloud are related through the conditions [11, (2.1) and (2.6)] required for rotational equilibrium. For our purpose, these conditions are recast in the form

$$\frac{1}{X^3\alpha} = Y - \frac{1}{2}\left(\frac{\omega_z}{\omega_c}\right)^2 Y^2, \tag{7}$$

$$X^3\alpha(\alpha^2-1) = Q_1^0\left(\frac{\alpha}{\sqrt{\alpha^2-1}}\right) \tag{8}$$

where $Q_1^0(\chi)$ is an associated Legendre polynomial of the second kind in χ. The first of these coupled equations, (7) is derived by taking the divergenece of the Lorentz force law and eliminating the electric field with Gauss' law. It has a centrifugal term $\propto Y^2$ in a frame corotating with the ions. In the cold fluid model, (8) represents the solution to Maxwell's equations for a uniformly charged spheroid rotating in an ideal Penning trap. It uses the electrostatic appoximation, wherein fields generated by the ion motions are negligible. Interaction with image charges induced in the trap electrodes is also negligible since the ions are assumed to be distributed in a central volume much smaller than the trap size [10], [11].

The dependence of the time-dilation shift on system parameters comes almost entirely from the mass-independent scaling factors S_1, ω_m, and b because (7) depends weakly on $(\omega_z/\omega_c)^2$ for small values of ω_z/ω_c, as illustrated in Fig. 1. Convergence is particularly rapid for low rotation frequencies (i.e., Y of order unity). In the limit $\omega_z/\omega_c \to 0$, the product X^2Y^2 yields a universal curve. This universal ($\omega_z/\omega_c \to 0$) limit is of experimental interest since it gives the smallest time-dilation shift and provides a good approximation for the experimental regime $\omega_z/\omega_c \lesssim 0.1$. For convenience, subsequent discussions employ this limit. The minimum shift occurs at $X = X_o \simeq 1.06$ (see Fig. 1) and is attained by preparing the ions in a spheroid with aspect ratio $\alpha = z_s/r_s \simeq 0.460$ rotating at $\omega_r \simeq 1.82\omega_m(\omega_c/\omega_c') \approx 1.82\omega_m$. To the extent that this limit is a good approximation, the optimal shape of the spheroid is independent of system parameters, and the fractional frequency shift near the minimum is

$$\frac{\Delta\nu}{\nu_o} \simeq -S_1\left[3.721 + 3.4\left(\frac{\Delta r_s}{r_{so}}\right)^2\right]. \tag{9}$$

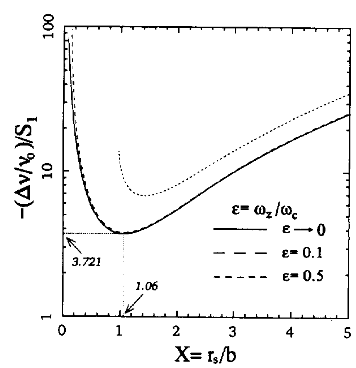

Fig. 1. Time-dilation shift versus spheroid radius, for various values of $\epsilon \equiv \omega_z/\omega_c$. Solid curve gives the universal limit as $\epsilon \to 0$.

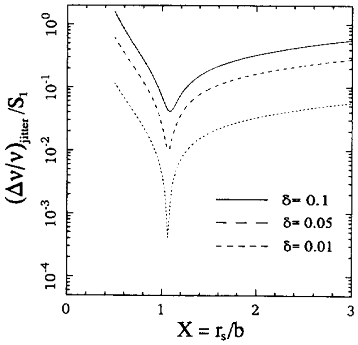

Fig. 2. Root-mean-squared frequency jitter in a single-species ion clock as a function of mean spheroid radius X, for various values of the Gaussian standard deviation δ in the radius fluctuations.

This is expressed in terms of the deviation Δr_s from the optimal spheroid radius $r_{so} = 1.06\ b$ because the spheroid radius is typically monitored in real-time with an imaging device and thus provides an observable for locking schemes to stabilize the spheroid. By stabilizing the spheroid radius near the optimal value r_{so}, it should be possible to reduce the jitter in the time-dilation shift. If the spheroid radius fluctuates about X with Gaussian standard deviation δ, the frequency jitters by

$$\left\langle\left(\frac{\Delta\nu}{\nu}\right)^2\right\rangle_{jitter} = \int_{-\infty}^{+\infty} dX' \frac{1}{\delta\sqrt{2\pi}} \exp\left[-\frac{(X'-X)^2}{2\delta^2}\right] \cdot \left[\frac{\nu(X') - \langle\nu\rangle}{\nu_o}\right]^2. \tag{10}$$

As shown in Fig. 2, near the minimum-shift configuration, even a modest stabilization of $\delta \lesssim 0.1$ (i.e., $(\Delta r_s)_{rms}/r_{so} \lesssim 0.1$) can reduce the frequency jitter to less than 1% of the time-dilation shift, i.e., $\Delta\nu_{rms}/\nu \lesssim 4 \times 10^{-2} S_1$.

In the $\omega_z/\omega_c \to 0$ limit, the minimum time-dilation shift ($\Delta\nu_{min}/\nu_o \simeq -3.721\ S_1$) is independent of the ion mass. Thus, ions with clock transitions which are field-independent at high magnetic fields are desirable since the scaling factor S_1

TABLE I

EXPECTED PERFORMANCE FOR A SINGLE-SPECIES ION CLOCK WITH $N = 10^6$, $T_R = 100$ s, $\alpha = 0.46$, AND $r_{so} = 4.2$ mm. EQUATION (1) IS USED TO EVALUATE σ_y. VALUES OF ν_o AND B_o ARE TAKEN FROM [12], [13]

Ion	ν_o (GHz)	B_o (T)	ω_z/ω_c	$\sigma_y(1\ \mathrm{s})$ $\times 10^{15}$	minimum $\Delta\nu/\nu_o$ $\times 10^{15}$
$^9\mathrm{Be}^+$	0.303016	0.8194	0.0989	53	−241
$^{25}\mathrm{Mg}^+$	0.291996	1.2398	0.1089	55	−105
$^{67}\mathrm{Zn}^+$	$\simeq 1$	$\simeq 8.0$	0.0276	16	−2.5
$^{199}\mathrm{Hg}^+$	20.9	43.9	0.0087	0.76	−0.084
$^{201}\mathrm{Hg}^+$	7.73	3.91	0.0979	2.1	−11

is inversely proportional to the square of the magnetic field. The expected performance for several systems with $N = 10^6$, $T_R = 100$ s, and fixed spheroid dimensions ($\alpha = 0.46$, $r_{so} = 4.2$ mm) is given in Table I. The gain in frequency stability with large numbers of ions will allow measurements of the time-dilation shift in reasonably short integration time. Since $S_1 \propto V_o^{4/3}$ (V_o is the trapping voltage), it should be possible, by measuring the minimally shifted clock frequency for various V_o, to extrapolate down to the $V_o = 0$ limit for the unshifted clock frequency to an accuracy limited by $\sigma_y(\tau)$. The slope of the extrapolation would provide an independent determination of the number of ions.

III. SYMPATHETICALLY COOLED ION CLOCK

In a frequency standard, it is desirable to interrogate the clock transition without the perturbations (a.c. Stark shifts) caused by the cooling laser. If the heating by the environment is sufficiently weak, then the cooling laser can be turned off during the interrogation cycles. However, when the interrogation time is rather long or heating is significant, continuous cooling of the clock ions is required. This cooling can be provided by using a second, simultaneously-stored laser-cooled ion species, which sympathetically cools the clock ions through the Coulomb coupling between species [14]–[16]. When the second ion species, the "coolant" ions, have smaller charge-to-mass ratio than the clock ions, the two species of ions centrifugally separate [14]–[16] with the coolant ions occupying an annular region surrounding a central core of clock ions. Ramsey interrogation times as long as $T_R \simeq 550$ s have been attained using laser-cooled $^{24}\mathrm{Mg}^+$ ions to sympathetically cool $^9\mathrm{Be}^+$ clock ions [3], [4]. Moreover, the time-dilation shift is smaller for sympathetically cooled clocks since the clock ions are distributed closer to the axis than in a single-species configuration.

For low rotation frequency and $\omega_z/\omega_c \ll 1$, the surface enclosing the two species may be approximated by the spheroidal surface for a single species. This is because, for a single species in this limit, the centrifugal term in (7) becomes negligible and the ion distribution becomes independent of ion mass to order $(\omega_z/\omega_c)^2$. Therefore, if a cylindrical core is removed from a minimum-shift configuration and is substituted by an equal number of clock ions, as illustrated in Fig. 3, the spheroidal boundary is well preserved. Neglecting the small gap [14]–[16] between the two species, (5) and (9) derived for the time-dilation shift in a single species, are still applicable to the cylindrical core of clock ions provided the scaling factor S_1 is replaced by

$$S_2 \equiv S_1' \Lambda \tag{11}$$

where S_1' is S_1 evaluated assuming $N = N_1 + N_2$ (N_1 is the number of clock ions and N_2 is the number of coolant ions) and Λ is defined by

$$\Lambda \equiv 1 - \frac{3}{2}\frac{N_2}{N_1}\left[1 - \left(\frac{N_2}{N_1 + N_2}\right)^{2/3}\right]. \tag{12}$$

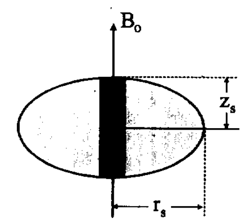

Fig. 3. Cross-sectional diagram of a two-species ion plasma for an ion clock. Coolant ions occupy the annular region surrounding the clock ions (shown darker) in the cylindrical core.

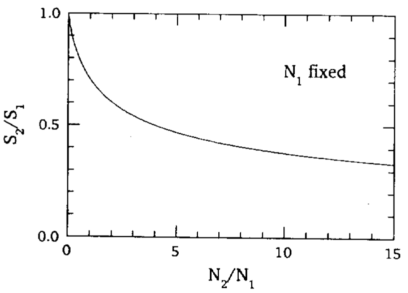

Fig. 4. Ratio of the time-dilation shift in a sympathetically cooled clock to that in a single-species clock. The number of clock ions is fixed.

Fig. 4 plots S_2/S_1 as a function of N_2/N_1, comparing the time-dilation shift in a two-species clock to that in a single-species clock with the same number of clock ions N_1. In the regime $N_2/N_1 \gg 1$, we have $S_2/S_1 \simeq (5/6)\sqrt[3]{N_1/N_2}$. For a typical $N_2/N_1 \sim 10$, the time-dilation shift is about 60% smaller than in a single-species clock.

The design and realization of an ion frequency standard, of course, must take into account other considerations [12]. For instance, preliminary studies showed that the $^9\mathrm{Be}^+$ hyperfine clock transition (303 MHz) has an unexpectedly large pressure shift $\{(-1.7 \pm 0.4) \times 10^{-5}\ \mathrm{Pa}^{-1}$ for $\mathrm{CH}_4\}$ [3], [4]. For some ions, a cryogenic environment may be required to suppress this effect. Significant improvement in both frequency stability and accuracy also requires the ability to trap and laser-cool large numbers of ions, as well as to characterize and control the ion distribution for minimizing the time-dilation shift. Experiments are underway in a new Penning trap [17] designed for these and other studies.

ACKNOWLEDGMENT

The authors thank A. Barton and S. Gilbert for carefully reading the manuscript.

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